Let $M(i,a)$ be the elementary matrix for multiplying row i by a.

Let $A(i,a,j)$ be the elementary matrix for adding a times row i to row j.

We start with

$$[A \quad I] = \left[\begin{array}{rrr|rrr} -2 & 3 & 2 & 1 & 0 & 0 \\ 6 & 0 & 3 & 0 & 1 & 0 \\ 4 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{aligned} & A(1,-4,3)A(1,-6,2)M\left(1,-\tfrac{1}{2}\right)[A \quad I] \\ = & \left[\begin{array}{rrr|rrr} 1 & -\frac{3}{2} & -1 & -\frac{1}{2} & 0 & 0 \\ 0 & 9 & 9 & 3 & 1 & 0 \\ 0 & 7 & 3 & 2 & 0 & 1 \end{array}\right] \\ = & [A_1 \quad B_1], \text{ say.} \end{aligned}$$

$$\begin{aligned} & A(2,-7,3)A\left(2,\tfrac{3}{2},1\right)M\left(2,\frac{1}{9}\right)[A_1 \quad B_1] \\ = & \left[\begin{array}{rrr|rrr} 1 & 0 & \frac{1}{2} & 0 & \frac{1}{6} & 0 \\ 0 & 1 & 1 & \frac{1}{3} & \frac{1}{9} & 0 \\ 0 & 0 & -4 & -\frac{1}{3} & -\frac{7}{9} & 1 \end{array}\right] \\ = & [A_2 \quad B_2], \text{ say.} \end{aligned}$$

$$\begin{aligned} & A(3,-1,2)A(3,-\tfrac{1}{2},1)M\left(3,-\tfrac{1}{4}\right)[A_2 \quad B_2] \\ = & \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -\frac{1}{24} & \frac{5}{72} & \frac{1}{8} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{12} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{12} & \frac{7}{36} & -\frac{1}{4} \end{array}\right] \\ = & [I \quad B], \text{ say.} \end{aligned}$$

Thus

$$\begin{aligned} & A(3,-1,2)A(3,-\tfrac{1}{2},1)M\left(3,-\tfrac{1}{4}\right) \\ \times \quad & A(2,-7,3)A(2,\tfrac{3}{2},1)M(2,\tfrac{1}{9}) \\ \times \quad & A(1,-4,3)A(1,-6,2)M(1,-\tfrac{1}{2})A \\ & = I. \end{aligned}$$

Hence

$$\begin{aligned} A &= [(3,-1,2)A(3,-\tfrac{1}{2},1)M\left(3,-\tfrac{1}{4}\right) \\ &\quad \times A(2,-7,3)A(2,\tfrac{3}{2},1)M(2,\tfrac{9}{)} \\ &\quad \times A(1,-4,3)A(1,-6,2)M(1,-\tfrac{1}{2})]^{-1}, \end{aligned}$$

or

$$\begin{aligned} A &= M(1,-2)A(1,6,2)A(1,4,3) \\ &\quad \times M(2,9)A(2,-\tfrac{3}{2},1)A(2,7,3) \\ &\quad \times M(3,-4)A(3,\tfrac{1}{2},1)A(3,1,2), \end{aligned}$$

since for any n nonsingular matrices $A_1, ..., A_n$ we have

$$(A_1 \cdots A_n)^{-1} = A_n^{-1} \cdots A_1^{-1},$$

and

$$M(i,a)^{-1} = M(i,\tfrac{1}{a}) \text{ and } A(i,a,j)^{-1} = A(i,-a,j).$$

Thus we have expressed A as a product of elementary matrices.

Also,

$$\begin{aligned} A^{-1} &= A(3,-1,2)A(3,-\tfrac{1}{2},1)M\left(3,-\tfrac{1}{4}\right) \\ &\quad \times A(2,-7,3)A(2,\tfrac{3}{2},1)M\left(2,\tfrac{1}{9}\right) \\ &\quad \times A(1,-4,3)A(1,-6,2)M\left(1,-\tfrac{1}{2}\right) \\ &= B. \end{aligned}$$

So the inverse is

$$A^{-1} = \begin{bmatrix} -\frac{1}{24} & \frac{5}{72} & \frac{1}{8} \\ \frac{1}{4} & -\frac{1}{12} & \frac{1}{4} \\ \frac{1}{12} & \frac{7}{36} & -\frac{1}{4} \end{bmatrix}.$$

লক্ষ কর যে A^{-1} বার করার কাজটা কিন্তু সহজই ছিল। A-কে elementary matrix দিয়ে ভেঙে লেখাটাই যা কঠিন। এই প্রশ্নে সেটা চেয়েছিল বলে করেছি। এই ভেঙে লেখাটা কিন্তু A^{-1} বার করতে কোথাও কাজে লাগে নি। ■

Example 20: Express the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 3 & 0 \\ 6 & 2 & 3 \end{bmatrix}$$

as a product of elementary matrices and hence find A^{-1}. **(2006)**

SOLUTION: