FUNCTIONAL DEPENDENCE

W. F. NEWNS, The University, Liverpool

1. Introduction. It has been the fashion for some time for textbooks on Advanced Calculus or Analysis to include a section on Functional Dependence, the general idea being as follows. Real-valued functions $f_1, f_2, \cdots, f_m$ of n real variables are functionally dependent if they satisfy a relation of the form $F(f_1(x), f_2(x), \cdots, f_m(x)) = 0$ identically. Assuming the f_i to be continuously differentiable, it is claimed that the functions are dependent if and only if the rank of their Jacobian is everywhere less than m. It is clearly not intended that every sequence of m functions shall be regarded as dependent (by taking F identically zero) but just what *is* intended by the definition is usually obscure. In the proof of 'only if' in the above result, it is usually supposed that F is continuously differentiable and that its partial derivatives nowhere vanish simultaneously. The 'if' statement is usually deduced from the implicit function theorem on the assumption that the Jacobian has constant rank, and appears to be a local result. From what is assumed about F in the proof of 'only if,' and proved about F in the converse result, one can arrive at an appropriate meaning of 'dependence' for this particular theorem. This meaning, however, differs from the definition given by R. C. Buck [2] in one of the few textbooks where the definition is precise. It also differs from the definition used by A. B. Brown [1], on the basis of which he proves a result similar to the textbook result, but without the hypothesis that the Jacobian has constant rank. (In Brown's theorem there was an extra differentiability hypothesis on the f_i, a hypothesis which was later weakened by A. Sard [9].)

In view of this, it seems worth while to attempt an exposition of the subject, starting with an investigation into the relationship between various possible meanings of 'dependence.'

2. Definitions. We first make

DEFINITION 1. *Let X, I be sets, $(Y_\iota)_{\iota \in I}$ a family of sets, and for each $\iota \in I$ let $f_\iota: X \to Y_\iota$. Let F be a real-valued function with domain $\mathfrak{D}(F) \subset \prod_{\iota \in I} Y_\iota$. Then we say that the f_ι are F-related iff $F(f(x)) = 0$ for all $x \in X$, where $f: X \to \prod_{\iota \in I} Y_\iota$ is the function having (f_ι) as components.*

It is implied here that if (f_ι) is F-related then $\mathfrak{D}(F)$ contains the range $\mathfrak{R}(f)$ of f.

To obtain a useful definition of dependent (as F-related for some suitable F) we must not merely exclude the case where F is identically zero: we need something like the other extreme. The obvious other extreme, namely that F has no zeros, is clearly inappropriate. In a topological space, however, a function is identically zero iff its support is empty. We recall the definition.

DEFINITION 2. *Let Y be a topological space and F a real-valued function with domain Y. The support of F is the subset*

$$\operatorname{supp} F = \overline{\{y \in Y: F(y) \neq 0\}}$$

of Y, where the bar denotes closure in Y.

The support of F is the smallest closed set outside which F vanishes identically. The other extreme from supp $F = \emptyset$ is supp $F = \mathfrak{D}(F)$, but since all we require of $\mathfrak{D}(F)$ is that it should contain $\mathfrak{R}(f)$, it is equivalent for our purposes to make the following definition, which specializes essentially to that given by R. C. Buck ([2] p. 226).

DEFINITION 3. *Let X, I be sets, (Y_ι) a family of topological spaces and for each $\iota \in \mathrm{I}$ let $f_\iota: X \rightarrow Y_\iota$. We say that the f_ι are functionally dependent iff they are F-related by some F such that supp $F \supset \mathfrak{R}(f)$.*

Buck's requirement on F, namely that it should vanish identically on no nonempty open set, is clearly equivalent.

Having ensured by our requirement on F that the relation satisfied by the dependent functions is as nontrivial as possible, it is natural to impose further restrictions on F.

DEFINITION 4. *Let X, I, (Y_ι), (f_ι) be as in Definition 3. Then we say that the f_ι are C^0-dependent iff they are F-related by some continuous F such that* supp $F \supset \mathfrak{R}(f)$.

DEFINITION 5. *Let X be a set, let m be a positive integer and for each integer i $(1 \leqq i \leqq m)$ let f_i be a real-valued function with domain X. Let p denote ∞ or a positive integer. Then the f_i are C^p-dependent iff they are F-related by some F of class C^p such that* supp $F \supset \mathfrak{R}(f)$. *They are analytically dependent iff they are F-related by some analytic F such that* supp $F \supset \mathfrak{R}(f)$.

3. Topological characterizations of dependence.

PROPOSITION 1. *Let X, I, (Y_ι), (f_ι), f be as in Definition 3. Then the f_ι are functionally dependent iff $\mathfrak{R}(f)$ has no interior point (in $Y = \pi Y_\iota$).*

Proof. Suppose the f_ι are functionally dependent and choose F relating them and such that supp $F \supset \mathfrak{R}(f)$. If y_0 were an interior point of $\mathfrak{R}(f)$, then $\mathfrak{R}(f)$ would be a neighborhood of y_0 on which F vanishes, hence not meeting $\{y \in \mathfrak{D}(F): F(y) \neq 0\}$. This would mean $y_0 \notin \operatorname{supp} F$ contrary to $y_0 \in \mathfrak{R}(f) \subset \operatorname{supp} F$. We conclude that no such point exists.

Conversely, suppose that $\mathfrak{R}(f)$ has no interior point and take F to be the characteristic function of $Y \setminus \mathfrak{R}(f)$. Then the f_ι are F-related, and the support of F is the closure of $Y \setminus \mathfrak{R}(f)$, whose complement is an open subset of $\mathfrak{R}(f)$, and hence, is empty. Thus supp $F = Y$ and the f_ι are functionally dependent.

PROPOSITION 2. *If the f_ι are C^0-dependent then $\mathfrak{R}(f)$ is nowhere dense in Y.*

Proof. Let F be a continuous function relating the f_ι and satisfying supp $F \supset \mathfrak{R}(f)$. Then F vanishes on $\overline{\mathfrak{R}(f)} \cap \mathfrak{D}(F)$ and *a fortiori* on $U \cap \mathfrak{D}(F)$, where U is the interior of $\overline{\mathcal{R}(f)}$: thus $U \cap \operatorname{supp} F = \varnothing$. If U were nonempty, it would meet $\overline{\mathcal{R}(f)}$ (it is a subset of it) and hence meet $\mathcal{R}(f)$ contrary to $\mathcal{R}(f) \subset \operatorname{supp} F$. We conclude that $U = \varnothing$.

A converse of Proposition 2 is not to be expected without some restriction on the topologies of the Y_ι. We prove only an easy result.

PROPOSITION 3. *Suppose that* I *is countable, that each* Y_ι *is metrizable and that* $\mathcal{R}(f)$ *is nowhere dense in* Y. *Then the components of* f *are* C^0*-dependent.*

Proof. The space Y being metrizable, we choose a metric for it and let $F(y)$ be the distance of y from $\mathcal{R}(f)$. Then F is continuous and $F(y) = 0$ iff $y \in \overline{\mathcal{R}(f)}$. Thus the components of f are F-related and since $\overline{\mathcal{R}(f)}$ has empty interior, supp $F = Y$.

For the case of C^p-dependence, since this implies C^0-dependence we know from Proposition 2 that a nowhere dense range is necessary for C^p-dependence. Conversely:

PROPOSITION 4. *Let* X *be a set,* m *a positive integer and* $f: X \to \boldsymbol{R}^m$ *a function whose range is nowhere dense in* $\boldsymbol{R}^m$. *Then the components of* f *are* C^∞*-dependent.*

Proof. For any integers n, k $(1 \leqq k \leqq m)$, the equation $y_k = n$ defines a hyperplane in $\boldsymbol{R}^m$ (y_k denoting the kth coordinate of $y \in \boldsymbol{R}^m$). All such hyperplanes divide $\boldsymbol{R}^m$ into open hypercubes, and we denote by Q_1 the set of all those hypercubes not meeting $\mathcal{R}(f)$. For any integer $s > 1$, we divide $\boldsymbol{R}^m$ similarly into hypercubes by means of hyperplanes with equations

$$y_k = n2^{-s} (n \in \boldsymbol{Z},\ 1 \leqq k \leqq m),$$

and denote by Q_s the set of those hypercubes which meet neither $\mathcal{R}(f)$ nor any element of Q_r for $r < s$. For any positive integer s, we put

$$F_s(y) = s^{-s} \exp\left(-(\sin(2^s \pi y_1) \sin(2^s \pi y_2) \cdots \sin(2^s \pi y_m))^{-2}\right)$$

whenever y belongs to some element of Q_s, and $F_s(y) = 0$ for all other y in $\boldsymbol{R}^m$. It can be shown that F_s is of class C^∞, hence so also is $F = \sum_{s=1}^\infty F_s$. It is clear that F vanishes on $\mathcal{R}(f)$, and hence that the components of f are F-related; also, the support of F contains the union of the Q_s, hence also the complement of $\overline{\mathcal{R}(f)}$, and since $\overline{\mathcal{R}(f)}$ has empty interior we see that supp $F = \boldsymbol{R}^m$.

REMARK. By a modified construction it is possible to prove: *Given any closed subset* S *of* $\boldsymbol{R}^m$, *there is an* F: $\boldsymbol{R}^m \to \boldsymbol{R}$ *of class* C^∞ *such that* $F(y) = 0$ *iff* $y \in S$. See [1] for details.

For the case of m real-valued functions we see that, for any p, C^p-dependence is equivalent to C^0-dependence and to C^∞-dependence. We thus have only three distinct kinds of dependence. (That the first two are distinct can be seen by considering the projection onto the (x, y)-plane of a well-known dense curve on a torus. Defining

$$f\colon \boldsymbol{R} \to \boldsymbol{R}^2 \text{ by } f(t) = e^{it}(2 + \cos(t\sqrt{2})),$$

we can show that $\mathcal{R}(f)$ has no interior point but is dense in the annulus $\{z \in \boldsymbol{C}\colon 1 \leqq |z| \leqq 3\}$. For analytic dependence, an extra topological condition on $\mathcal{R}(f)$ is necessary [**1**].)

4. Local dependence. We now suppose that X is a topological space. A family (f_ι) is *locally C^0-dependent* iff each point of X has a fundamental system of neighborhoods such that the restrictions of the f_ι to each of the neighborhoods are C^0-dependent. It is clearly enough that each point of X should possess one such neighborhood. Moreover,

PROPOSITION 5. *If Y is metrizable and the components of f are locally C^0-dependent, then for every compact $K \subset X$, the components of $f|_K$ are C^0-dependent.*

Proof. Each point of K has a neighborhood U such that $f|_U$ has C^0-dependent components, hence such that $f[U]$ is nowhere dense in Y (Prop. 2). We can cover K by a finite sequence (U_i) of such U. Thus $f[K]$, as subset of $\bigcup f[U_i]$, is nowhere dense, and consequently (Prop. 3) the components of $f|_K$ are C^0-dependent.

We next show that, at least in the most usual case, this kind of dependence is not yet another kind, but is equivalent to the first kind. The generality is retained mainly to indicate what is involved.

PROPOSITION 6. *Suppose that X is locally compact and σ-compact, that Y is a complete metric space and that f is continuous. Then the components of f are functionally dependent iff they are locally C^0-dependent.*

REMARK. The definition of functional dependence used by A. B. Brown [**1**] is essentially (i.e., modulo the irrelevance of p in C^p-dependence) the property given in Proposition 5, and so (since $\boldsymbol{R}^n$ is locally compact) is equivalent to our local C^0-dependence. Proposition 6 asserts the equivalence of Brown's definition with Buck's.

Proof of Proposition 6. Suppose that the components of f are functionally dependent. Then $\mathcal{R}(f)$ has no interior point. Let $x \in X$ and choose a compact neighborhood K of x. *A fortiori*, $f[K]$ has no interior point, and being closed (as compact subset of a Hausdorff space) is nowhere dense. By Proposition 3, the components of $f|_K$ are C^0-dependent. Thus f has locally C^0-dependent components.

Conversely, supposing that f has locally C^0-dependent components, let (K_n) be a sequence of compact sets covering X. By Proposition 5, the components of $f|_{K_n}$ are C^0-dependent, hence (Proposition 2) $f[K_n]$ is nowhere dense, for every n. The sets $f[K_n]$ cover $\mathcal{R}(f)$, so that $\mathcal{R}(f)$ is meager (1st. category). Since Y is a Baire space (2nd. category), $\mathcal{R}(f)$ has no interior point. The result follows from Proposition 1.

COROLLARY. *Let m, n be positive integers, $X \subset \boldsymbol{R}^n$ and let $f\colon X \to \boldsymbol{R}^m$ be continuous and have functionally dependent components. Then for any compact $K \subset X$ the components of $f|_K$ are C^∞-dependent.*

5. Functional dependence and vanishing Jacobians. The basic result of Brown and Sard mentioned in the introduction is as follows:

THEOREM 1. *Let m, n be positive integers, X an open subset of $\boldsymbol{R}^n$ and $f\colon X\to\boldsymbol{R}^m$ of class C^p for some $p\geqq 1$. Then*

(i) *if the components of f are functionally dependent, their Jacobian has rank less than m at every point of X;*

(ii) *the converse of* (i) *is true whenever $p\geqq\frac{1}{2}(n-m+2)$ (hence without class restrictions if $m\geqq n$).*

Before considering the proof of this theorem, we recall that the *differential* df_a of f is the linear function $u\colon\boldsymbol{R}^n\to\boldsymbol{R}^m$ (unique when it exists) such that

$$\lim_{x\to a}\frac{f(x)-f(a)-u(x-a)}{\|x-a\|}=0,$$

where $\|\ \|$ denotes some norm on $\boldsymbol{R}^n$. Which norm is used is irrelevant since all norms on $\boldsymbol{R}^n$ are equivalent. The one most frequently used is the *Euclidean norm*, determined by $\|x\|^2=\sum_{j=1}^n x_j^2$, where the x_j are the coordinates of x. We shall also use the *Cartesian norm*, defined by

$$\|x\|=\sup\{\,|x_j|:1\leqq j\leqq n\}.$$

The continuity of df_a (which is automatic for a linear mapping between finite-dimensional spaces) is expressed by the existence of $M\in\boldsymbol{R}$ such that $\|df_a(x)\|\leqq M\|x\|$ for all $x\in\boldsymbol{R}^n$. The infimum of all such M has the same property and is denoted by $\|df_a\|$. The partial derivatives of the components of f at a are the elements of the Jacobian matrix of f at a, which is the matrix of the linear function df_a relative to the natural bases for $\boldsymbol{R}^n$ and $\boldsymbol{R}^m$. The rank of this Jacobian matrix is equal to the rank of the differential.

In the sequel, we use the following result, called by Dieudonné [**3**, p. 273] the

RANK THEOREM. *Let m, n, X, f, p be as in Theorem* 1, *and suppose given a point $a\in\mathfrak{D}(f)$ such that df_x is of constant rank r for all x in some neighborhood of a. Then there exist neighborhoods U_1, U_2 of 0, a in $\boldsymbol{R}^n$, neighborhoods V_1, V_2 of $f(a)$, 0 in $\boldsymbol{R}^m$ and functions $g\colon U_1\to U_2$, $h\colon V_1\to V_2$ with the following properties:*

(i) *$g(0)=a$, $h(f(a))=0$ and g, h are one-one and have ranges U_2, V_2 respectively,*

(ii) *g, g^{-1}, h, h^{-1} are of class C^p,*

(iii) *$h\circ f\circ g=df_{a|U_1}$,*

Moreover,

(iv) *if $r=n$ (in which case it is enough to be given that df_a has rank n, since then df_x will have rank n on some neighborhood of a: equivalently if df_a is one-one) then g can be chosen to be a translation in $\boldsymbol{R}^n$,*

(v) *if $r=m$ (equivalently, if df_a has range $\boldsymbol{R}^m$), then h can be chosen to be a translation in $\boldsymbol{R}^m$.*

Readers unfamiliar with this coordinate-free treatment are recommended to consult chapter 10 of [**4**].

Proof of (i). Suppose that, for some point $a \in X$, the differential df_a of f at a has rank m. Then we find from the rank theorem that $f[U_2] = h^{-1}[df_a[U_1]]$ is a neighborhood of $f(a)$, so that $f(a)$ is an interior point of $\mathcal{R}(f)$. The assertion now follows from Proposition 1.

The proof of (ii) will not be given in full. We shall indicate what is involved, and supply references.

It is enough to prove that if K is a closed cube in X with sides parallel to the axes, then $f[K]$ has no interior point. The technique is to show that $f[K]$ lies in a set of arbitrarily small volume, from which it is obvious that it cannot have an interior point.

In fact, this proves more, namely that $f[K]$ has measure zero, where 'measure' means either Lebesgue measure in $\boldsymbol{R}^m$ or m-dimensional Hausdorff measure. We use a cube, because this can be subdivided into nonoverlapping cubes. For convenience, we use the Cartesian norm on $\boldsymbol{R}^n$, so that the cubes are 'balls' in the sense of this metric. On $\boldsymbol{R}^m$ we use the Euclidean norm since we wish to estimate volume.

According to the Mean Value theorem,

$$\|f(x) - f(a)\| \leqq M\|x - a\| \tag{1}$$

for all $x \in K$, $a \in K$, where $M = \sup\{\|df_x\| : x \in K\}$ is finite. In other words, distances are increased by a factor of at most M. Let Q be a cube in K with sides of length l parallel to the axes. Then $f[Q]$ lies inside a ball of radius $\frac{1}{2}Ml$, hence of volume $M_1 l^m$, where M_1 is independent of Q. Letting L denote the length of the sides of K and dividing K into N^n cubes with sides of length L/N, we see that $f[K]$ lies in a set of volume not exceeding $N^n M_1 (L/N)^m$ which is arbitrarily small for large enough N provided $m > n$. Thus the result is proved for this case.

If $m \leqq n$ we make more specific use of the hypothesis that the rank of the Jacobian is everywhere less than m. From the compactness of K, it follows that there exists an increasing function $b: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ with $\lim_{\epsilon \to 0} b(\epsilon) = 0$ such that

$$\|f(x) - f(a) - df_a(x - a)\| \leqq b(\|x - a\|)\|x - a\| \tag{2}$$

whenever $x \in K$ and $a \in K$. For fixed a, the points $f(a) + df_a(x - a)$ all lie in a hyperplane H in $\boldsymbol{R}^m$ since $\mathcal{R}(df_a)$ has dimension less than m. The above inequality shows that a cube of side l maps to a set of points no further than $lb(l)$ from H, hence into a hypercylinder of volume $M_2 l^{m-1} lb(l)$. Subdividing into N^n cubes of side L/N as above, we see that $f[K]$ lies in a set of volume $M_2 L^m N^{n-m} b(L/N)$, which is now arbitrarily small for large N when $m = n$.

The way to cope with the case $m < n$ is to modify the factor N^{n-m} in the estimate. We need to improve (1) to

$$\|f(x) - f(a)\| \leqq M\|x - a\|^q \tag{3}$$

so that a cube of side l maps into a ball of radius $M(l/2)^q$ and (with account taken of the rank of df) into a hypercylinder of volume $M_3 l^{q(m-1)} lb(l)$. This would give the estimate $M_3L^{qm-q+1}N^{n-q(m-1)-1}b(L/N)$ which is good enough provided $n-q(m-1)-1\leqq 0$, i.e., $q\geqq(n-1)/(m-1)$, but an inequality of the form (3) for $q>1$ cannot be expected to hold very widely. Clearly, if it holds for all x in some neighborhood of a point a, we must have $df_a=0$. Moreover, when q is an integer, it suggests that the Taylor expansion of f about a has all terms of orders $1, 2, \cdots, q-1$ equal to 0, and that $M\|x-a\|^q$ estimates the remainder term. We expect therefore to need to assume f to be of class C^q as well as to consider points a at which $df_a=0$. Of course, if $df_x=0$ for all x in a ball, then f will be constant on that ball, and the image of the ball gives no trouble: our problem is to cope with $\{x\in X: df_x=0\}$, a closed set A which may be extremely complicated. Before stating the precise result, we observe that the isolated points of any subset of $\boldsymbol{R}^n$ are countable (and so have countable image), whilst the nonisolated points of A are accumulation points of points where df vanishes, and we may therefore expect a zero of higher order at such points.

LEMMA 1. *Let $A\subset X\subset\boldsymbol{R}^n$, X being open, and let p be a positive integer. Then there exists a sequence $(A_i)_{i\in\boldsymbol{N}}$ of subsets of X such that*: (i) $A\subset\bigcup_{i\in\boldsymbol{N}}A_i$, (ii) *$A_0$ is countable*, (iii) *For any $f: X\to\boldsymbol{R}$ of class C^p whose differential vanishes on A and any $i>0$, there exists an increasing function $b_i: \boldsymbol{R}_+\to\boldsymbol{R}_+$ such that $\lim_{\epsilon\to 0}b_i(\epsilon)=0$ and*

$$|f(x)-f(a)|\leqq b_i(\|x-a\|)\|x-a\|^p$$

whenever $x\in A_i$ and $a\in A_i$.

From this, as already indicated, we deduce

LEMMA 2. *Let X be an open subset of $\boldsymbol{R}^n$, let $f: X\to\boldsymbol{R}^m$ be of class C^p, and let $A=\{x\in X: df_x=0\}$. Then $f[A]$ is of measure zero, provided $p\geqq n/m$.*

The slight reduction (from $(n-1)/(m-1)$ to n/m) comes about because the M in (3) has become a $b_i(\|x-a\|)$ and we can therefore consider balls rather than hypercylinders. From this we obtain:

LEMMA 3. *Let X be an open subset of $\boldsymbol{R}^n$, let $f: X\to\boldsymbol{R}^m$ be of class C^p and let A_r be the set of points $x\in X$ at which the rank of df_x is $r(<m)$. Then $f[A_r]$ has measure zero provided that $p\geqq(n-r)/(m-r)$.*

For $r=0$ this is Lemma 2. For a point a where df_a has rank $r>0$, we let X_2 be the kernel of df_a, Y_1 the range of df_a and choose supplementary subspaces X_1, Y_2 in $\boldsymbol{R}^n$, $\boldsymbol{R}^m$ respectively. After a C^p change of coordinates (by applying the rank theorem to $\pi_1\circ f$, where π_1 is projection onto Y_1) we find that the restriction of the new function f^* (which is defined on a neighborhood N of 0) to any slice of N (formed by taking the intersection with N of some translate of X_2) has two properties: its range lies in the appropriate translate of Y_2 and its differential vanishes at any point corresponding (by the coordinate change) to a point of A_r. The dimensions of X_2, Y_2 being $n-r$, $m-r$ respectively, Lemma 2 applies to such restrictions to show that $f[A_r]$ meets every translate of Y_2 in a set of $(m-r)$-dimensional measure zero, and the result follows from Fubini's Theorem.

When the rank of df_x is less than m for all $x\in X$, we have $\mathcal{R}(f)=\bigcup_{r=0}^{m-1} f[A_r]$ and so is of measure 0 (hence without interior point) provided $p\geqq(n-r)/(m-r)$ for $r=0, 1, \cdots, m-1$, i.e., provided $p\geqq n-m+1$. The slight improvement in part (ii) of Theorem 1 comes from the observation that Lemma 3 is needed only for $r\leqq m-2$. That $f[A_{m-1}]$ has no interior point is immediate from the rank theorem, A_{m-1} being an open set on which the differential of f has constant rank $m-1$.

References. Lemma 1 was given in [6] Theorem 4.2. Lemmas 2 and 3 were given in [9]. There is also a full discussion of Sard's theorem, including Morse's lemma, in [10]. Sard's theorem is the corollary of Lemma 3 (without supposing the rank of the differential to be everywhere less than m) that the set $\bigcup_{r=2}^{m-1} f[A_r]$ has measure 0 provided $p\geqq n-m+1$. It is used in the study of immersions of manifolds and in the case $m=n$ is relevant to the change of variable in a multiple integral (cf. [5]).

REMARKS. 1. Although part (ii) of Theorem 1 is deduced from Sard's results, these results were designed to prove Sard's Theorem and do not use the hypothesis of Theorem 1 (ii) that every point is a critical point. Thus, although the lower bound on p in Sard's theorem cannot be improved (as shown by an example in [11]), that in Theorem 1 (ii) is obviously not the best possible for the (admittedly trivial) case $m=1$. Presumably a new approach is needed.

2. Let m, n, X, f and p be as in Theorem 1. Suppose that $a\in X$ and $r<m$ are such that df_x has rank r for all x in some neighborhood of a. Then we can find neighbourhoods U, V of a, $b(=f(a))$ respectively and $F\colon V\to \mathbf{R}^{m-r}$ of class C^p such that $f[U]\subset V\subset \mathbf{R}^m$, $F\circ f|_U=0$ and dF_a has rank $m-r$. (We have only to let π_2 be projection onto a supplement of the range of df_a and take $U=U_2$, $V=V_1$, $F=\pi_2\circ h$ in the notation of the rank theorem.) The components of F therefore constitute $m-r$ independent relations between the components of $f|_U$: and these relations can be solved to give $m-r$ of the components of f in terms of the others. This is the usual textbook result, and it is used in the discussion of total differential equations (cf. [8] p. 141, for example).

It should be noted that without the hypothesis of constant rank these results may fail. For example, let $f=(f_1, f_2)\colon \mathbf{R}\to\mathbf{R}^2$ be defined by

$$f_1(t) = \begin{cases} e^{-1/t^2} & (t>0) \\ 0 & (t\leqq 0), \end{cases}$$

$f_2(t)=f_1(-t)$. Then f_1 and f_2 are analytically dependent (take $F(x, y)=xy$), but are not F-related on a neighborhood of the origin by any F of class C^1 for which dF_0 has rank 1. Nor is there any ϕ such that $f_1(t)=\phi(f_2(t))$ on some neighborhood of 0.

6. Analytic dependence. There is no general analogue of Theorem 1 above for analytic dependence, as is shown by the following example (cf. [**7**]).

Define $f: \mathbf{C}^2 \to \mathbf{C}^m$ for $m \geqq 3$ by

$$f(u, v) = (u, uv, uve_1(v), uve_2(u), \cdots, uve_{m-2}(v)),$$

where $e_1(v) = e^v$ and $e_k(v) = e^v e_{k-1}(e^v)$. Let F be a complex-valued analytic function on some neighborhood of 0 in $\mathbf{C}^m$ such that $F \circ f$ vanishes on some neighborhood of 0 in $\mathbf{C}^2$. We shall prove that F vanishes identically.

Collecting terms in the power series expansion of F about the origin, we can write

$$F = \sum_{n=0}^{\infty} F_n,$$

where F_n is a homogeneous polynomial function of total degree n in m variables. Composing with f, we have

$$\begin{aligned} F(f(u, v)) &= \sum_{n=0}^{\infty} F_n(u, uv, uve_1(v), \cdots, uve_{m-2}(v)) \\ &= \sum_{n=0}^{\infty} u^n F_n(1, v, ve_1(v), \cdots, ve_{m-2}(v)), \end{aligned}$$

a power series in u with coefficients depending on v. Since this vanishes on a neighborhood of 0 in $\mathbf{C}^2$, we conclude that the coefficient of each power of u vanishes on a neighborhood of 0 in $\mathbf{C}$, hence everywhere in $\mathbf{C}$ (since it is an entire function). Writing

$$F_n(y_1, y_2, \cdots, y_m) = \sum_{k=0}^{n} F_{nk}(y_2, y_3, \cdots, y_m) y_1^{n-k},$$

the F_{nk} are homogeneous polynomial functions of total degree k in $m-1$ variables and $\sum_{k=0}^{n} F_{nk}(v, ve_1(v), \cdots, ve_{m-2}(v)) = 0$ for all $v \in \mathbf{C}$. Hence

$$\sum_{k=0}^{n} v^k F_{nk}(1, e_1(v), \cdots, e_{m-2}(v)) = 0$$

for all $v \in \mathbf{C}$. Let $v_0 \in \mathbf{C}$. Then the polynomial function

$$v \mapsto \sum_{k=0}^{n} v^k F_{nk}(1, e_1(v_0), \cdots, e_{m-2}(v_0))$$

vanishes for $v = v_0 + 2q\pi i$ for all $q \in \mathbf{Z}$, since e_k takes the same value at all such v. We conclude that it vanishes identically and since v_0 is arbitrary, that

$$F_{nk}(1, e_1(v), \cdots, e_{m-2}(v)) = 0$$

for all $v \in \mathbf{C}$. If we write $w = e^v$, this means that

$$F_{nk}(1, w, we_1(w), \cdots, we_{m-3}(w)) = 0$$

for all $w \in C \setminus \{0\}$, hence for all $w \in C$. But this is precisely what we had earlier, but with $m-1$ for m. It is therefore enough to consider the case $m=3$, for which we have

$$F_{nk}(1, w) = 0$$

for all $w \in C$. This gives $F_{nk}=0$ immediately, and completes the proof that $F=0$.

By defining g: $C^n \to C^m$ by $g(x_1, x_2, \cdots, x_n) = f(x_1, x_2)$ we see that:

If $n \geqq 2$ and $m \geqq 3$, the components of an analytic f: $C^n \to C^m$ whose differential has rank everywhere less than m may fail to be locally analytically dependent.

For the remaining cases, the positive result does hold. The case $m=1$ is trivial. The case $m=2$, was proved by A. B. Brown in [1]. The case $n=1$, $m>2$ follows from the case $m=2$ by using the first two components.

For the real analytic case, the same discussion as above gives the analogous negative results. The positive results follow from the complex case.

References

1. A. B. Brown, Functional dependence, Trans. Amer. Math. Soc., 38 (1935) 379–394.
2. R. C. Buck, Advanced Calculus, McGraw-Hill, New York, 1956.
3. J. Dieudonné, Foundations of Modern Analysis, Academic Press, New York, 1960.
4. T. M. Flett, Mathematical Analysis, McGraw-Hill, London, 1966.
5. ———, On transformation in $\mathbf{R}^n$ and a theorem of Sard, this MONTHLY, 71 (1964) 623–629.
6. A. P. Morse, The behavior of a function on its critical set, Ann. of Math., (2) 40 (1939) 62–70.
7. W. F. Osgood, On functions of several complex variables, Trans. Amer. Math. Soc., 17 (1916) 1–8.
8. H. T. H. Piaggio, Differential Equations, Bell, London, 1945.
9. A. Sard, The measure of the critical values of differentiable maps, Bull. Amer. Math. Soc., 48 (1942) 883–890.
10. S. Sternberg, Lectures on Differential Geometry, Prentice-Hall, Englewood Cliffs, N. J., 1964.
11. H. Whitney, A function not constant on a connected set of critical points, Duke Math. J., 1 (1935) 514–17.

BOUNDS ON FUNCTIONS OF MATRICES

P. C. ROSENBLOOM, University of Minnesota and Minnesota State Department of Education

1. Introduction. If f is a function on a set S of scalars (real or complex) to the scalars, then the postulate, if $Ax = \lambda x$ and $\lambda \in S$, then

$$f(A)x = f(\lambda)x \tag{1}$$

uniquely defines the extension of f to the set S_n of all $n \times n$ matrices with distinct eigenvalues belonging to S. For various classes of functions f and sets S formulas