Chapter 1

Limits and Continuity

1.1. The Limit of a Function

We adopt the following definitions.

Definition 1. A real function f is said to be *increasing* (resp. *strictly increasing, decreasing, strictly decreasing*) on a nonempty set $\mathbf{A} \subset \mathbb{R}$ if $x_1 < x_2$, $x_1, x_2 \in \mathbf{A}$, implies $f(x_1) \leq f(x_2)$ (resp. $f(x_1) < f(x_2), f(x_1) \geq f(x_2), f(x_1) > f(x_2)$). A function which is either increasing or decreasing (resp. strictly increasing or strictly decreasing) is called *monotone* (resp. *strictly monotone*).

Definition 2. By a *deleted neighborhood* of a point $a \in \mathbb{R}$ we mean the set $(a - \varepsilon, a + \varepsilon) \setminus \{a\}$, where $\varepsilon > 0$.

1.1.1. Find the limits or state that they do not exist.

(a) $\displaystyle\lim_{x\to 0} x\cos\frac{1}{x}$, (b) $\displaystyle\lim_{x\to 0} x\left[\frac{1}{x}\right]$,

(c) $\displaystyle\lim_{x\to 0} \frac{x}{a}\left[\frac{b}{x}\right]$, $a, b > 0$, (d) $\displaystyle\lim_{x\to 0} \frac{[x]}{x}$,

(e) $\displaystyle\lim_{x\to +\infty} x\left(\sqrt{x^2+1} - \sqrt[3]{x^3+1}\right)$, (f) $\displaystyle\lim_{x\to 0} \frac{\cos\left(\frac{\pi}{2}\cos x\right)}{\sin(\sin x)}$.

1.1.2. Assume that $f:(-a,a)\setminus\{0\}\to\mathbb{R}$. Show that

(a) $\lim_{x\to0} f(x)=l$ if and only if $\lim_{x\to0} f(\sin x)=l$,

(b) if $\lim_{x\to0} f(x)=l$, then $\lim_{x\to0} f(|x|)=l$. Does the other implication hold?

1.1.3. Suppose a function $f:(-a,a)\setminus\{0\}\to(0,+\infty)$ satisfies $\lim_{x\to0}\left(f(x)+\frac{1}{f(x)}\right)=2$. Show that $\lim_{x\to0} f(x)=1$.

1.1.4. Assume f is defined on a deleted neighborhood of a and $\lim_{x\to a}\left(f(x)+\frac{1}{|f(x)|}\right)=0$. Determine $\lim_{x\to a} f(x)$.

1.1.5. Prove that if f is a bounded function on $[0,1]$ satisfying $f(ax)=bf(x)$ for $0\le x\le\frac{1}{a}$ and $a,b>1$, then $\lim_{x\to0^+} f(x)=f(0)$.

1.1.6. Calculate

(a) $$\lim_{x\to0}\left(x^2\left(1+2+3+\cdots+\left[\frac{1}{|x|}\right]\right)\right),$$

(b) $$\lim_{x\to0^+}\left(x\left(\left[\frac{1}{x}\right]+\left[\frac{2}{x}\right]+\cdots+\left[\frac{k}{x}\right]\right)\right),\quad k\in\mathbb{N}.$$

1.1.7. Compute $\lim_{x\to\infty}\frac{P([x])}{P(x)}$, where $P(x)$ is a polynomial with positive coefficients.

1.1.8. Show by an example that the condition

$$(*)\qquad \lim_{x\to0}(f(x)+f(2x))=0$$

does not imply that f has a limit at 0. Prove that if there exists a function φ such that in a deleted neighborhood of zero the inequality $f(x)\ge\varphi(x)$ is satisfied and $\lim_{x\to0}\varphi(x)=0$, then $(*)$ implies $\lim_{x\to0} f(x)=0$.

1.1.9.

(a) Give an example of a function f satisfying the condition

$$\lim_{x\to0}(f(x)f(2x))=0$$

and such that $\lim_{x\to 0} f(x)$ does not exist.

(b) Show that if in a deleted neighborhood of zero the inequalities $f(x) \ge |x|^\alpha$, $\frac{1}{2} < \alpha < 1$, and $f(x)f(2x) \le |x|$ hold, then $\lim_{x\to 0} f(x) = 0$.

1.1.10. Given a real α, assume that $\lim_{x\to\infty} \frac{f(ax)}{x^\alpha} = g(a)$ for each positive a. Show that there is c such that $g(a) = ca^\alpha$.

1.1.11. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a monotonic function such that $\lim_{x\to\infty} \frac{f(2x)}{f(x)} = 1$. Show that also $\lim_{x\to\infty} \frac{f(cx)}{f(x)} = 1$ for each $c > 0$.

1.1.12. Prove that if $a > 1$ and $\alpha \in \mathbb{R}$, then

(a) $\displaystyle\lim_{x\to\infty} \frac{a^x}{x} = +\infty,$ (b) $\displaystyle\lim_{x\to\infty} \frac{a^x}{x^\alpha} = +\infty.$

1.1.13. Show that if $\alpha > 0$, then $\lim_{x\to\infty} \frac{\ln x}{x^\alpha} = 0$.

1.1.14. For $a > 0$, show that $\lim_{x\to 0} a^x = 1$. Use this equality to prove the continuity of the exponential function.

1.1.15. Show that

(a) $\displaystyle\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e,$ (b) $\displaystyle\lim_{x\to-\infty} \left(1 + \frac{1}{x}\right)^x = e,$

(c) $\lim_{x\to 0} (1 + x)^{\frac{1}{x}} = e.$

1.1.16. Show that $\lim_{x\to 0} \ln(1+x) = 0$. Using this equality, deduce that the logarithmic function is continuous on $(0, \infty)$.

1.1.17. Determine the following limits:

(a) $\displaystyle\lim_{x\to 0} \frac{\ln(1 + x)}{x},$ (b) $\displaystyle\lim_{x\to 0} \frac{a^x - 1}{x},\ a > 0,$

(c) $\displaystyle\lim_{x\to 0} \frac{(1 + x)^\alpha - 1}{x},\ \alpha \in \mathbb{R}.$

1.1.18. Find

(a) $\lim\limits_{x\to\infty}(\ln x)^{\frac{1}{x}}$, (b) $\lim\limits_{x\to0^+} x^{\sin x}$,

(c) $\lim\limits_{x\to0}(\cos x)^{\frac{1}{\sin^2 x}}$, (d) $\lim\limits_{x\to\infty}(e^x-1)^{\frac{1}{x}}$,

(e) $\lim\limits_{x\to0^+}(\sin x)^{\frac{1}{\ln x}}$.

1.1.19. Find the following limits:

(a) $\lim\limits_{x\to0}\dfrac{\sin 2x+2\arctan 3x+3x^2}{\ln(1+3x+\sin^2 x)+xe^x}$, (b) $\lim\limits_{x\to0}\dfrac{\ln\cos x}{\tan x^2}$,

(c) $\lim\limits_{x\to0^+}\dfrac{\sqrt{1-e^{-x}}-\sqrt{1-\cos x}}{\sqrt{\sin x}}$, (d) $\lim\limits_{x\to0}(1+x^2)^{\cot x}$.

1.1.20. Calculate

(a) $\lim\limits_{x\to\infty}\left(\tan\dfrac{\pi x}{2x+1}\right)^{\frac{1}{x}}$, (b) $\lim\limits_{x\to\infty} x\left(\ln\left(1+\dfrac{x}{2}\right)-\ln\dfrac{x}{2}\right)$.

1.1.21. Suppose that $\lim\limits_{x\to0^+} g(x)=0$ and that there are $\alpha\in\mathbb{R}$ and positive m and M such that $m\le\frac{f(x)}{x^\alpha}\le M$ for positive x from a neighborhood of zero. Show that if $\alpha\lim\limits_{x\to0^+} g(x)\ln x=\gamma$, then $\lim\limits_{x\to0^+} f(x)^{g(x)}=e^\gamma$. In the case where $\gamma=\infty$ or $\gamma=-\infty$ we assume that $e^\infty=\infty$ and $e^{-\infty}=0$.

1.1.22. Assume that $\lim\limits_{x\to0} f(x)=1$ and $\lim\limits_{x\to0} g(x)=\infty$. Show that if $\lim\limits_{x\to0} g(x)(f(x)-1)=\gamma$, then $\lim\limits_{x\to0} f(x)^{g(x)}=e^\gamma$.

1.1.23. Calculate

(a) $\lim\limits_{x\to0^+}\left(2\sin\sqrt{x}+\sqrt{x}\sin\dfrac{1}{x}\right)^x$,

(b) $\lim\limits_{x\to0}\left(1+xe^{-\frac{1}{x^2}}\sin\dfrac{1}{x^4}\right)^{e^{\frac{1}{x^2}}}$,

(c) $\lim\limits_{x\to0}\left(1+e^{-\frac{1}{x^2}}\arctan\dfrac{1}{x^2}+xe^{-\frac{1}{x^2}}\sin\dfrac{1}{x^4}\right)^{e^{\frac{1}{x^2}}}$.

1.1.24. Let $f : [0, +\infty) \to \mathbb{R}$ be a function such that each sequence $\{f(a+n)\}$, $a \geq 0$, converges to zero. Does the limit $\lim\limits_{x\to\infty} f(x)$ exist?

1.1.25. Let $f : [0, +\infty) \to \mathbb{R}$ be a function such that, for any positive a, the sequence $\{f(an)\}$ converges to zero. Does the limit $\lim\limits_{x\to\infty} f(x)$ exist?

1.1.26. Let $f : [0, +\infty) \to \mathbb{R}$ be a function such that, for each $a \geq 0$ and each $b > 0$, the sequence $\{f(a+bn)\}$ converges to zero. Does the limit $\lim\limits_{x\to+\infty} f(x)$ exist?

1.1.27. Prove that if $\lim\limits_{x\to 0} f(x) = 0$ and $\lim\limits_{x\to 0} \frac{f(2x)-f(x)}{x} = 0$, then $\lim\limits_{x\to 0} \frac{f(x)}{x} = 0$.

1.1.28. Suppose that f defined on $(a, +\infty)$ is bounded on each finite interval (a, b), $a < b$. Prove that if $\lim\limits_{x\to+\infty} (f(x+1) - f(x)) = l$, then also $\lim\limits_{x\to+\infty} \frac{f(x)}{x} = l$.

1.1.29. Let f defined on $(a, +\infty)$ be bounded below on each finite interval (a, b), $a < b$. Show that if $\lim\limits_{x\to+\infty} (f(x+1) - f(x)) = +\infty$, then also $\lim\limits_{x\to+\infty} \frac{f(x)}{x} = +\infty$.

1.1.30. Let f defined on $(a, +\infty)$ be bounded on each finite interval (a, b), $a < b$. If for a nonnegative integer k, $\lim\limits_{x\to+\infty} \frac{f(x+1)-f(x)}{x^k}$ exists, then

$$\lim_{x\to+\infty} \frac{f(x)}{x^{k+1}} = \frac{1}{k+1} \lim_{x\to+\infty} \frac{f(x+1) - f(x)}{x^k}.$$

1.1.31. Let f defined on $(a, +\infty)$ be bounded on each finite interval (a, b), $a < b$, and assume that $f(x) \geq c > 0$ for $x \in (a, +\infty)$. Show that if $\lim\limits_{x\to+\infty} \frac{f(x+1)}{f(x)}$ exists, then $\lim\limits_{x\to+\infty} (f(x))^{\frac{1}{x}}$ also exists and

$$\lim_{x\to+\infty} (f(x))^{\frac{1}{x}} = \lim_{x\to+\infty} \frac{f(x+1)}{f(x)}.$$

1.1.32. Assume that $\lim\limits_{x\to 0} f\left(\left[\frac{1}{x}\right]^{-1}\right) = 0$. Does this imply that the limit $\lim\limits_{x\to 0} f(x)$ exists?

1.1.33. Let $f : \mathbb{R} \to \mathbb{R}$ be such that, for any $a \in \mathbb{R}$, the sequence $\left\{f\left(\frac{a}{n}\right)\right\}$ converges to zero. Does f have a limit at zero?

1.1.34. Prove that if $\lim\limits_{x\to 0} f\left(x\left(\frac{1}{x} - \left[\frac{1}{x}\right]\right)\right) = 0$, then $\lim\limits_{x\to 0} f(x) = 0$.

1.1.35. Show that if f is monotonically increasing (decreasing) on (a, b), then for any $x_0 \in (a, b)$,

(a) $$f(x_0^+) = \lim_{x\to x_0^+} f(x) = \inf_{x>x_0} f(x) \quad \left(f(x_0^+) = \sup_{x>x_0} f(x)\right),$$

(b) $$f(x_0^-) = \lim_{x\to x_0^-} f(x) = \sup_{x<x_0} f(x) \quad \left(f(x_0^-) = \inf_{x<x_0} f(x)\right),$$

(c) $$f(x_0^-) \le f(x_0) \le f(x_0^+) \qquad \left(f(x_0^-) \ge f(x_0) \ge f(x_0^+)\right).$$

1.1.36. Show that if f is monotonically increasing on (a, b), then for any $x_0 \in (a, b)$,

(a) $$\lim_{x\to x_0^+} f(x^-) = f(x_0^+),$$

(b) $$\lim_{x\to x_0^-} f(x^+) = f(x_0^-).$$

1.1.37. Prove the following *Cauchy theorem*. In order that f have a finite limit when x tends to a, a necessary and sufficient condition is that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(x')| < \varepsilon$ whenever $0 < |x - a| < \delta$ and $0 < |x' - a| < \delta$. Formulate and prove an analogous necessary and sufficient condition in order that $\lim\limits_{x\to\infty} f(x)$ exist.

1.1.38. Show that if $\lim\limits_{x\to a} f(x) = A$ and $\lim\limits_{y\to A} g(y) = B$, then $\lim\limits_{x\to a} g(f(x)) = B$, provided $(g \circ f)(x) = g(f(x))$ is well defined and f does not attain A in a deleted neighborhood of a.

1.1.39. Find functions f and g such that $\lim_{x\to a} f(x) = A$ and $\lim_{y\to A} g(y) = B$, but $\lim_{x\to a} g(f(x)) \neq B$.

1.1.40. Suppose $f: \mathbb{R} \to \mathbb{R}$ is an increasing function and $x \mapsto f(x) - x$ has the period 1. Denote by f^n the nth *iterate of* f; that is, $f^1 = f$ and $f^n = f \circ f^{n-1}$ for $n \geq 2$. Prove that if $\lim_{n\to\infty} \frac{f^n(0)}{n}$ exists, then for every $x \in \mathbb{R}$, $\lim_{n\to\infty} \frac{f^n(x)}{n} = \lim_{n\to\infty} \frac{f^n(0)}{n}$.

1.1.41. Suppose $f: \mathbb{R} \to \mathbb{R}$ is an increasing function and $x \mapsto f(x) - x$ has the period 1. Moreover, suppose that $f(0) > 0$ and p is a fixed positive integer. Let f^n denote the nth iterate of f. Prove that if m_p is the least positive integer such that $f^{m_p}(0) > p$, then

$$\frac{p}{m_p} \leq \varliminf_{n\to\infty} \frac{f^n(0)}{n} \leq \varlimsup_{n\to\infty} \frac{f^n(0)}{n} \leq \frac{p}{m_p} + \frac{1+f(0)}{m_p}.$$

1.1.42. Suppose $f: \mathbb{R} \to \mathbb{R}$ is an increasing function and $x \mapsto f(x) - x$ has the period 1. Show that $\lim_{n\to\infty} \frac{f^n(x)}{n}$ exists and its value is the same for each $x \in \mathbb{R}$, where f^n denotes the nth iterate of f.

1.2. Properties of Continuous Functions

1.2.1. Find all points of continuity of f defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ irrational,} \\ \sin|x| & \text{if } x \text{ is rational.} \end{cases}$$

1.2.2. Determine the set of points of continuity of f given by

$$f(x) = \begin{cases} x^2 - 1 & \text{if } x \text{ is irrational,} \\ 0 & \text{if } x \text{ is rational.} \end{cases}$$

1.2.3. Study the continuity of the following functions:

(a) $$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational or } x = 0, \\ 1/q & \text{if } x = p/q,\ p \in \mathbb{Z},\ q \in \mathbb{N}, \text{ and} \\ & p, q \text{ are co-prime,} \end{cases}$$

(b) $$f(x) = \begin{cases} |x| & \text{if } x \text{ is irrational or } x = 0, \\ qx/(q+1) & \text{if } x = p/q,\ p \in \mathbb{Z},\ q \in \mathbb{N}, \text{ and} \\ & p, q \text{ are co-prime.} \end{cases}$$

(The function defined in (a) is called *the Riemann function.*)

1.2.4. Prove that if $f \in C([a,b])$, then $|f| \in C([a,b])$. Show by an example that the converse is not true.

1.2.5. Determine all a_n and b_n for which the function defined by

$$f(x) = \begin{cases} a_n + \sin \pi x & \text{if } x \in [2n, 2n+1],\ n \in \mathbb{Z}, \\ b_n + \cos \pi x & \text{if } x \in (2n-1, 2n),\ n \in \mathbb{Z}, \end{cases}$$

is continuous on $\mathbb{R}$.

1.2.6. Let $f(x) = [x^2] \sin \pi x$ for $x \in \mathbb{R}$. Study the continuity of f.

1.2.7. Let

$$f(x) = [x] + (x - [x])^{[x]} \quad \text{for} \quad x \geq \frac{1}{2}.$$

Show that f is continuous and that it is strictly increasing on $[1, \infty)$.

1.2.8. Study the continuity of the following functions and sketch their graphs:

(a) $$f(x) = \lim_{n \to \infty} \frac{n^x - n^{-x}}{n^x + n^{-x}}, \quad x \in \mathbb{R},$$

(b) $$f(x) = \lim_{n \to \infty} \frac{x^2 e^{nx} + x}{e^{nx} + 1}, \quad x \in \mathbb{R},$$

(c) $$f(x) = \lim_{n \to \infty} \frac{\ln(e^n + x^n)}{n}, \quad x \geq 0,$$

(d) $$f(x) = \lim_{n \to \infty} \sqrt[n]{4^n + x^{2n} + \frac{1}{x^{2n}}}, \quad x \neq 0,$$

(e) $$f(x) = \lim_{n \to \infty} \sqrt[2n]{\cos^{2n} x + \sin^{2n} x}, \quad x \in \mathbb{R}.$$

1.2.9. Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and periodic, then it attains its supremum and infimum.

1.2.10. For $P(x) = x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1 x + a_0$, show that there is $x_* \in \mathbb{R}$ such that $P(x_*) = \inf\{P(x) : x \in \mathbb{R}\}$. Show also that the absolute value of any polynomial P attains its infimum; that is, there is $x^* \in \mathbb{R}$ such that $|P(x^*)| = \inf\{|P(x)| : x \in \mathbb{R}\}$.

1.2.11.

(a) Give an example of a bounded function on $[0,1]$ which achieves neither an infimum nor a supremum.

(b) Give an example of a bounded function on $[0,1]$ which does not achieve its infimum on any $[a,b] \subset [0,1]$, $a < b$.

1.2.12. For $f : \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$ and $\delta > 0$, set

$$\omega_f(x_0, \delta) = \sup\{|f(x) - f(x_0)| : x \in \mathbb{R},\ |x - x_0| < \delta\}$$

and $\omega_f(x_0) = \lim\limits_{\delta \to 0^+} \omega_f(x_0, \delta)$. Show that f is continuous at x_0 if and only if $\omega_f(x_0) = 0$.

1.2.13.

(a) Let $f, g \in C([a,b])$ and for $x \in [a,b]$ let $h(x) = \min\{f(x),\ g(x)\}$ and $H(x) = \max\{f(x),\ g(x)\}$. Show that $h, H \in C([a,b])$.

(b) Let $f_1, f_2, f_3 \in C([a,b])$ and for $x \in [a,b]$ let $f(x)$ denote that one of the three values $f_1(x), f_2(x)$ and $f_3(x)$ that lies between the other two. Show that $f \in C([a,b])$.

1.2.14. Prove that if $f \in C([a,b])$, then the functions defined by setting

$$m(x) = \inf\{f(\zeta) : \zeta \in [a,x]\} \quad \text{and} \quad M(x) = \sup\{f(\zeta) : \zeta \in [a,x]\}$$

are also continuous on $[a,b]$.

1.2.15. Let f be a bounded function on $[a,b]$. Show that the functions defined by

$$m(x) = \inf\{f(\zeta) : \zeta \in [a,x)\} \quad \text{and} \quad M(x) = \sup\{f(\zeta) : \zeta \in [a,x)\}$$

are continuous from the left on (a,b).

1.2.16. Verify whether under the assumptions of the foregoing problem the functions

$$m^*(x)=\inf\{f(\zeta):\ \zeta\in[a,x]\}\quad\text{and}\quad M^*(x)=\sup\{f(\zeta):\ \zeta\in[a,x]\}$$

are continuous from the left on (a,b).

1.2.17. Suppose f is continuous on $[a,\infty)$ and $\lim\limits_{x\to\infty} f(x)$ is finite. Show that f is bounded on $[a,\infty)$.

1.2.18. Let f be continuous on $\mathbb{R}$ and let $\{x_n\}$ be a bounded sequence. Do the equalities

$$\varliminf_{n\to\infty} f(x_n) = f(\varliminf_{n\to\infty} x_n)\quad\text{and}\quad \varlimsup_{n\to\infty} f(x_n) = f(\varlimsup_{n\to\infty} x_n)$$

hold?

1.2.19. Let $f:\mathbb{R}\to\mathbb{R}$ be increasing and continuous and let $\{x_n\}$ be a bounded sequence. Show that

(a) $$\varliminf_{n\to\infty} f(x_n) = f(\varliminf_{n\to\infty} x_n),$$

(b) $$\varlimsup_{n\to\infty} f(x_n) = f(\varlimsup_{n\to\infty} x_n).$$

1.2.20. Let $f:\mathbb{R}\to\mathbb{R}$ be decreasing and continuous and let $\{x_n\}$ be a bounded sequence. Show that

(a) $$\varliminf_{n\to\infty} f(x_n) = f(\varlimsup_{n\to\infty} x_n),$$

(b) $$\varlimsup_{n\to\infty} f(x_n) = f(\varliminf_{n\to\infty} x_n).$$

1.2.21. Suppose that f is continuous on $\mathbb{R}$, $\lim\limits_{x\to-\infty} f(x) = -\infty$ and $\lim\limits_{x\to\infty} f(x) = +\infty$. Define g by setting

$$g(x) = \sup\{t:\ f(t) < x\}\quad\text{for}\quad x\in\mathbb{R}.$$

(a) Prove that g is continuous from the left.

(b) Is g continuous?

1.2.22. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous periodic function with two *incommensurate* periods T_1 and T_2; that is, $\frac{T_1}{T_2}$ is irrational. Prove that f is a constant function. Give an example of a nonconstant periodic function with two incommensurate periods.

1.2.23.

(a) Show that if $f : \mathbb{R} \to \mathbb{R}$ is nonconstant, periodic and continuous, then it has a smallest positive period, the so-called *fundamental period*.

(b) Give an example of a nonconstant periodic function without a fundamental period.

(c) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is a periodic function without a fundamental period, then the set of all periods of f is dense in $\mathbb{R}$.

1.2.24.

(a) Prove that the theorem in part (a) of the preceding problem remains true when the continuity of f on $\mathbb{R}$ is replaced by the continuity at one point.

(b) Show that if $f : \mathbb{R} \to \mathbb{R}$ is a periodic function without a fundamental period and if it is continuous at least at one point, then it is constant.

1.2.25. Show that if $f, g : \mathbb{R} \to \mathbb{R}$ are continuous and periodic and $\lim\limits_{x\to\infty}(f(x) - g(x)) = 0$, then $f = g$.

1.2.26. Give an example of two periodic functions f and g such that any period of f is not commensurate with any period of g and such that $f + g$

(a) is not periodic,

(b) is periodic.

1.2.27. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous and periodic with positive fundamental periods T_1 and T_2, respectively. Prove that if $\frac{T_1}{T_2} \notin \mathbb{Q}$, then $h = f + g$ is not a periodic function.

1.2.28. Let $f, g : \mathbb{R} \to \mathbb{R}$ be periodic and suppose that f is continuous and no period of g is commensurate with the fundamental period of f. Prove that $f + g$ is not a periodic function.

1.2.29. Prove that the set of points of discontinuity of a monotonic function $f : \mathbb{R} \to \mathbb{R}$ is at most countable.

1.2.30. Suppose f is continuous on $[0, 1]$. Prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (-1)^k f\left(\frac{k}{n}\right) = 0.$$

1.2.31. Let f be continuous on $[0, 1]$. Prove that

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f\left(\frac{k}{n}\right) = 0.$$

1.2.32. Suppose $f : (0, \infty) \to \mathbb{R}$ is a continuous function such that $f(x) \le f(nx)$ for all positive x and natural n. Show that $\lim\limits_{x\to\infty} f(x)$ exists (finite or infinite).

1.2.33. A function f defined on an interval $\mathbf{I} \subset \mathbb{R}$ is said to be *convex* on $\mathbf{I}$ if

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda)f(x_2)$$

whenever $x_1, x_2 \in \mathbf{I}$ and $\lambda \in (0, 1)$. Prove that if f is convex on an open interval, then it is continuous. Must a convex function on an arbitrary interval be continuous?

1.2.34. Prove that if a sequence $\{f_n\}$ of continuous functions on $\mathbf{A}$ converges uniformly to f on $\mathbf{A}$, then f is continuous on $\mathbf{A}$.

1.3. Intermediate Value Property

Recall the following:

Definition. A real function f has the *intermediate value property* on an interval $\mathbf{I}$ containing $[a, b]$ if $f(a) < v < f(b)$ or $f(b) < v < f(a)$;

that is, if v is between $f(a)$ and $f(b)$, there is between a and b a c such that $f(c) = v$.

1.3.1. Give examples of functions which have the intermediate value property on an interval **I** but are not continuous on this interval.

1.3.2. Prove that a strictly increasing function $f : [a, b] \to \mathbb{R}$ which has the intermediate value property is continuous on $[a, b]$.

1.3.3. Let $f : [0, 1] \to [0, 1]$ be continuous. Show that f has a *fixed point* in $[0, 1]$; that is, there exists $x_0 \in [0, 1]$ such that $f(x_0) = x_0$.

1.3.4. Assume that $f, g : [a, b] \to \mathbb{R}$ are continuous and such that $f(a) < g(a)$ and $f(b) > g(b)$. Prove that there exists $x_0 \in (a, b)$ for which $f(x_0) = g(x_0)$.

1.3.5. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and periodic with period $T > 0$. Prove that there is x_0 such that

$$f\left(x_0 + \frac{T}{2}\right) = f(x_0).$$

1.3.6. A function $f : (a, b) \to \mathbb{R}$ is continuous. Prove that, given $x_1, x_2, \ldots, x_n$ in (a, b), there exists $x_0 \in (a, b)$ such that

$$f(x_0) = \frac{1}{n}\left(f(x_1) + f(x_2) + \cdots + f(x_n)\right).$$

1.3.7.

(a) Prove that the equation $(1 - x)\cos x = \sin x$ has at least one solution in $(0, 1)$.

(b) For a nonzero polynomial P, show that the equation $|P(x)| = e^x$ has at least one solution.

1.3.8. For $a_0 < b_0 < a_1 < b_1 < \cdots < a_n < b_n$, show that all roots of the polynomial

$$P(x) = \prod_{k=0}^{n}(x + a_k) + 2\prod_{k=0}^{n}(x + b_k), \quad x \in \mathbb{R},$$

are real.

1.3.9. Suppose that f and g have the intermediate value property on $[a, b]$. Must $f + g$ possess the intermediate value property on that interval?

1.3.10. Assume that $f \in C([0,2])$ and $f(0) = f(2)$. Prove that there exist x_1 and x_2 in $[0,2]$ such that

$$x_2 - x_1 = 1 \quad \text{and} \quad f(x_2) = f(x_1).$$

Give a geometric interpretation of this fact.

1.3.11. Let $f \in C([0,2])$. Show that there are x_1 and x_2 in $[0,2]$ such that

$$x_2 - x_1 = 1 \quad \text{and} \quad f(x_2) - f(x_1) = \frac{1}{2}(f(2) - f(0)).$$

1.3.12. For $n \in \mathbb{N}$, let $f \in C([0,n])$ be such that $f(0) = f(n)$. Prove that there are x_1 and x_2 in $[0,n]$ satisfying

$$x_2 - x_1 = 1 \quad \text{and} \quad f(x_2) = f(x_1).$$

1.3.13. A continuous function f on $[0,n]$, $n \in \mathbb{N}$, satisfies $f(0) = f(n)$. Show that for every $k \in \{1, 2, \ldots, n-1\}$ there are x_k and x'_k such that $f(x_k) = f(x'_k)$, where $x_k - x'_k = k$ or $x_k - x'_k = n - k$. Is it true that for every $k \in \{1, 2, \ldots, n-1\}$ there are x_k and x'_k such that $f(x_k) = f(x'_k)$, where $x_k - x'_k = k$?

1.3.14. For $n \in \mathbb{N}$, let $f \in C([0,n])$ be such that $f(0) = f(n)$. Prove that the equation $f(x) = f(y)$ has at least n solutions with $x - y \in \mathbb{N}$.

1.3.15. Suppose that real continuous functions f and g defined on $\mathbb{R}$ commute; that is, $f(g(x)) = g(f(x))$ for $x \in \mathbb{R}$. Prove that if the equation $f^2(x) = g^2(x)$ has a solution, then the equation $f(x) = g(x)$ also has (here $f^2(x) = f(f(x))$ and $g^2(x) = g(g(x))$).

Show by example that the assumption of continuity of f and g in the foregoing problem cannot be omitted.

1.3.16. Prove that a continuous injection $f : \mathbb{R} \to \mathbb{R}$ is either strictly decreasing or strictly increasing.

1.3.17. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous injection. Prove that if there exists n such that the nth iteration of f is an identity, that is, $f^n(x) = x$ for all $x \in \mathbb{R}$, then

(a) $f(x) = x$, $x \in \mathbb{R}$, if f is strictly increasing,

(b) $f^2(x) = x$, $x \in \mathbb{R}$, if f is strictly decreasing.

1.3.18. Assume $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition $f(f(x)) = f^2(x) = -x$, $x \in \mathbb{R}$. Show that f cannot be continuous.

1.3.19. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which have the intermediate value property and such that there is $n \in \mathbb{N}$ for which $f^n(x) = -x$, $x \in \mathbb{R}$, where f^n denotes the nth iteration of f.

1.3.20. Prove that if $f : \mathbb{R} \to \mathbb{R}$ has the intermediate value property and $f^{-1}(\{q\})$ is closed for every rational q, then f is continuous.

1.3.21. Assume that $f : (a, \infty) \to \mathbb{R}$ is continuous and bounded. Prove that, given T, there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} x_n = +\infty \quad \text{and} \quad \lim_{n\to\infty} (f(x_n + T) - f(x_n)) = 0.$$

1.3.22. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ which attains each of its values exactly three times. Does there exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ which attains each of its values exactly two times?

1.3.23. Let $f : [0, 1] \to \mathbb{R}$ be continuous and *piecewise strictly monotone*. (A function f is said to be piecewise strictly monotone on $[0, 1]$, if there exists a partition of $[0, 1]$ into finitely many subintervals $[t_{i-1}, t_i]$, where $i = 1, 2, \dots, n$ and $0 = t_0 < t_1 < \dots < t_n = 1$, such that f is strictly monotone on each of these subintervals.) Prove that f attains at least one of its values an odd number of times.

1.3.24. A continuous function $f : [0, 1] \to \mathbb{R}$ attains each of its values finitely many times and $f(0) \neq f(1)$. Show that f attains at least one of its values an odd number of times.

1.3.25. Assume that $f : \mathbf{K} \to \mathbf{K}$ is continuous on a compact set $\mathbf{K} \subset \mathbb{R}$. Moreover, assume that an $x_0 \in \mathbf{K}$ is such that each limit point of the sequence of iterates $\{f^n(x_0)\}$ is a fixed point of f. Prove that $\{f^n(x_0)\}$ is convergent.

1.3.26. A function $f : \mathbb{R} \to \mathbb{R}$ is increasing, continuous, and such that F defined by $F(x) = f(x) - x$ is periodic with period 1. Prove that if $\alpha(f) = \lim\limits_{n\to\infty} \frac{f^n(0)}{n}$, then there is $x_0 \in [0,1]$ such that $F(x_0) = \alpha(f)$. Prove also that f has a fixed point in $[0,1]$ if and only if $\alpha(f) = 0$. (See Problems 1.1.40 - 1.1.42.)

1.3.27. A function $f : [0,1] \to \mathbb{R}$ satisfies $f(0) < 0$ and $f(1) > 0$, and there exists a function g continuous on $[0,1]$ and such that $f + g$ is decreasing. Prove that the equation $f(x) = 0$ has a solution in the open interval $(0,1)$.

1.3.28. Show that every bijection $f : \mathbb{R} \to [0,\infty)$ has infinitely many points of discontinuity.

1.3.29. Recall that each $x \in (0,1)$ can be represented by a binary fraction $.a_1a_2a_3\ldots$, where $a_i \in \{0,1\}$, $i = 1,2,\ldots$. In the case where x has two distinct binary expansions we choose the one with infinitely many digits equal to 1. Next let a function $f : (0,1) \to [0,1]$ be defined by

$$f(x) = \varlimsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} a_i.$$

Prove that f is discontinuous at each $x \in (0,1)$ but nevertheless it has the intermediate value property.

1.4. Semicontinuous Functions

Definition 1. The *extended real number system* $\overline{\mathbb{R}}$ consists of the real number system to which two symbols, $+\infty$ and $-\infty$, have been adjoined, with the following properties:

(i) If x is real, then $-\infty < x < +\infty$, and $x + \infty = +\infty$, $x - \infty = -\infty$ and $\frac{x}{+\infty} = \frac{x}{-\infty} = 0$.

(ii) If $x > 0$, then $x \cdot (+\infty) = +\infty$, $x \cdot (-\infty) = -\infty$.

(iii) If $x < 0$, then $x \cdot (+\infty) = -\infty$, $x \cdot (-\infty) = +\infty$.

Definition 2. If $\mathbf{A} \subset \overline{\mathbb{R}}$ is a nonempty set, then $\sup \mathbf{A}$ (resp. $\inf \mathbf{A}$) is the smallest (resp. greatest) extended real number which is greater (resp. smaller) than or equal to each element of $\mathbf{A}$.

Let f be a real-valued function defined on a nonempty set $\mathbf{A} \subset$.

Definition 3. If x_0 is a limit point of $\mathbf{A}$, then the *limit inferior* (resp. the *limit superior*) of $f(x)$ as $x \to x_0$ is defined as the infimum (resp. the supremum) of the set of all $y \in \overline{\mathbb{R}}$ such that there is a sequence $\{x_n\}$ of points in $\mathbf{A}$ which is convergent to x_0, whose terms are all different from x_0 and $y = \lim\limits_{n\to\infty} f(x_n)$. The limit inferior and the limit superior of $f(x)$ as $x \to x_0$ are denoted by $\varliminf\limits_{x\to x_0} f(x)$ and $\varlimsup\limits_{x\to x_0} f(x)$, respectively.

Definition 4. A real-valued function is said to be *lower* (resp. *upper*) *semicontinuous* at an $x_0 \in \mathbf{A}$ which is a limit point of $\mathbf{A}$ if $\varliminf\limits_{x\to x_0} f(x) \geq f(x_0)$ (resp. $\varlimsup\limits_{x\to x_0} f(x) \leq f(x_0)$). If x_0 is an isolated point of $\mathbf{A}$, then we assume that f is lower and upper semicontinuous at that point.

1.4.1. Show that if x_0 is a limit point of $\mathbf{A}$ and $f : \mathbf{A} \to \mathbb{R}$, then

(a) $$\varliminf_{x\to x_0} f(x) = \sup_{\delta>0} \inf\{f(x) :\ x \in \mathbf{A},\ 0 < |x - x_0| < \delta\},$$

(b) $$\varlimsup_{x\to x_0} f(x) = \inf_{\delta>0} \sup\{f(x) :\ x \in \mathbf{A},\ 0 < |x - x_0| < \delta\}.$$

1.4.2. Show that if x_0 is a limit point of $\mathbf{A}$ and $f : \mathbf{A} \to \mathbb{R}$, then

(a) $$\varliminf_{x\to x_0} f(x) = \lim_{\delta\to 0^+} \inf\{f(x) :\ x \in \mathbf{A},\ 0 < |x - x_0| < \delta\},$$

(b) $$\varlimsup_{x\to x_0} f(x) = \lim_{\delta\to 0^+} \sup\{f(x) :\ x \in \mathbf{A},\ 0 < |x - x_0| < \delta\}.$$

1.4.3. Prove that $y_0 \in \mathbb{R}$ is the limit inferior of $f : \mathbf{A} \to \mathbb{R}$ at a limit point x_0 of $\mathbf{A}$ if and only if for every $\varepsilon > 0$ the following two conditions are satisfied:

(i) there is $\delta > 0$ such that $f(x) > y_0 - \varepsilon$ for all $x \in \mathbf{A}$ with $0 < |x - x_0| < \delta$,

(ii) for every $\delta > 0$ there is $x' \in \mathbf{A}$ such that $0 < |x' - x_0| < \delta$ and $f(x') < y_0 + \varepsilon$.

Establish an analogous statement for the limit superior of f at x_0.

1.4.4. Let $f : \mathbf{A} \to \mathbb{R}$ and let x_0 be a limit point of $\mathbf{A}$. Prove that

(a) $\varliminf\limits_{x\to x_0} f(x) = -\infty$ if and only if for any real y and for any $\delta > 0$ there exists $x' \in \mathbf{A}$ such that $0 < |x' - x_0| < \delta$ and $f(x') < y$.

(b) $\varlimsup\limits_{x\to x_0} f(x) = +\infty$ if and only if for any real y and for any $\delta > 0$ there exists $x' \in \mathbf{A}$ such that $0 < |x' - x_0| < \delta$ and $f(x') > y$.

1.4.5. Suppose $f : \mathbf{A} \to \mathbb{R}$ and x_0 is a limit point of $\mathbf{A}$. Show that if $l = \varliminf\limits_{x\to x_0} f(x)$ (resp. $L = \varlimsup\limits_{x\to x_0} f(x)$), then there is a sequence $\{x_n\}$, $x_n \in \mathbf{A}$, $x_n \neq x_0$, converging to x_0 such that $l = \lim\limits_{n\to\infty} f(x_n)$ (resp. $L = \lim\limits_{n\to\infty} f(x_n)$).

1.4.6. Let $f : \mathbf{A} \to \mathbb{R}$ and let x_0 be a limit point of $\mathbf{A}$. Prove that

$$\varliminf_{x\to x_0} (-f(x)) = -\varlimsup_{x\to x_0} f(x) \quad \text{and} \quad \varlimsup_{x\to x_0} (-f(x)) = -\varliminf_{x\to x_0} f(x).$$

1.4.7. Let $f : \mathbf{A} \to (0, \infty)$ and let x_0 be a limit point of $\mathbf{A}$. Show that

$$\varliminf_{x\to x_0} \frac{1}{f(x)} = \frac{1}{\varlimsup\limits_{x\to x_0} f(x)} \quad \text{and} \quad \varlimsup_{x\to x_0} \frac{1}{f(x)} = \frac{1}{\varliminf\limits_{x\to x_0} f(x)}.$$

(We assume that $\frac{1}{+\infty} = 0$ and $\frac{1}{0^+} = +\infty$.)

1.4.8. Assume that $f, g : \mathbf{A} \to \mathbb{R}$ and that x_0 is a limit point of $\mathbf{A}$. Prove that (excluding the indeterminate forms of the type $+\infty - \infty$

and $-\infty+\infty$) the following inequalities hold:

$$\begin{aligned}\varliminf_{x\to x_0} f(x)+\varliminf_{x\to x_0} g(x) &\le \varliminf_{x\to x_0}(f(x)+g(x)) \le \varliminf_{x\to x_0} f(x)+\varlimsup_{x\to x_0} g(x)\\ &\le \varlimsup_{x\to x_0}(f(x)+g(x)) \le \varlimsup_{x\to x_0} f(x)+\varlimsup_{x\to x_0} g(x).\end{aligned}$$

Give examples of functions for which "$\le$ " in the above inequalities is replaced by "$<$ ".

1.4.9. Assume that $f, g : \mathbf{A} \to [0,\infty)$ and that x_0 is a limit point of $\mathbf{A}$. Prove that (excluding the indeterminate forms of the type $0\cdot(+\infty)$ and $(+\infty)\cdot 0$) the following inequalities hold:

$$\begin{aligned}\varliminf_{x\to x_0} f(x)\cdot\varliminf_{x\to x_0} g(x) &\le \varliminf_{x\to x_0}(f(x)\cdot g(x)) \le \varliminf_{x\to x_0} f(x)\cdot\varlimsup_{x\to x_0} g(x)\\ &\le \varlimsup_{x\to x_0}(f(x)\cdot g(x)) \le \varlimsup_{x\to x_0} f(x)\cdot\varlimsup_{x\to x_0} g(x).\end{aligned}$$

Give examples of functions for which "$\le$ " in the above inequalities is replaced by "$<$".

1.4.10. Prove that if $\lim\limits_{x\to x_0} f(x)$ exists, then (excluding the indeterminate forms of the type $+\infty-\infty$ and $-\infty+\infty$)

$$\begin{aligned}\varliminf_{x\to x_0}(f(x)+g(x)) &= \lim_{x\to x_0} f(x)+\varliminf_{x\to x_0} g(x),\\ \varlimsup_{x\to x_0}(f(x)+g(x)) &= \lim_{x\to x_0} f(x)+\varlimsup_{x\to x_0} g(x).\end{aligned}$$

Moreover, if f and g are nonnegative, then (excluding the indeterminate forms of the type $0\cdot(+\infty)$ and $(+\infty)\cdot 0$)

$$\begin{aligned}\varliminf_{x\to x_0}(f(x)\cdot g(x)) &= \lim_{x\to x_0} f(x)\cdot\varliminf_{x\to x_0} g(x),\\ \varlimsup_{x\to x_0}(f(x)\cdot g(x)) &= \lim_{x\to x_0} f(x)\cdot\varlimsup_{x\to x_0} g(x).\end{aligned}$$

1.4.11. Prove that if f is continuous on (a,b), $l = \varliminf\limits_{x\to a} f(x)$ and $L = \varlimsup\limits_{x\to a} f(x)$, then for every $\lambda\in[l,L]$ there is a sequence $\{x_n\}$ of points in (a,b) converging to a and such that $\lim\limits_{n\to\infty} f(x_n)=\lambda$.

1.4.12. Find the points at which $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ \sin x & \text{if } x \text{ is rational} \end{cases}$$

is semicontinuous.

1.4.13. Determine points at which the function f defined by

$$f(x) = \begin{cases} x^2 - 1 & \text{if } x \text{ is irrational,} \\ 0 & \text{if } x \text{ is rational} \end{cases}$$

is semicontinuous.

1.4.14. Show that the function given by setting

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational or } x = 0, \\ \frac{1}{q} & \text{if } x = \frac{p}{q},\ p \in \mathbb{Z}, q \in \mathbb{N}, \\ & \text{ and } p, q \text{ are co-prime} \end{cases}$$

is upper semicontinuous.

1.4.15. Find the points at which the function defined by

(a)
$$f(x) = \begin{cases} |x| & \text{if } x \text{ is irrational or } x = 0, \\ \frac{qx}{q+1} & \text{if } x = \frac{p}{q},\ p \in \mathbb{Z},\ q \in \mathbb{N}, \\ & \text{and } p, q \text{ are co-prime,} \end{cases}$$

(b)
$$f(x) = \begin{cases} \frac{(-1)^q p}{q+1} & \text{if } x \in \mathbb{Q} \cap (0,1] \text{ and } x = \frac{p}{q},\ p, q \in \mathbb{N}, \\ & \text{and } p, q \text{ are co-prime,} \\ 0 & \text{if } x \in (0,1) \text{ is irrational} \end{cases}$$

is neither upper nor lower semicontinuous.

1.4.16. Let $f, g : \mathbf{A} \to \mathbb{R}$ be lower (resp. upper) semicontinuous at $x_0 \in \mathbf{A}$. Show that

(a) if $a > 0$, then af is lower (resp. upper) semicontinuous at x_0. If $a < 0$, then af is upper (resp. lower) semicontinuous at x_0.

(b) $f + g$ is lower (resp. upper) semicontinuous at x_0.

1.4.17. Assume that $f_n : \mathbf{A} \to \mathbb{R}$, $n \in \mathbb{N}$, are lower (resp. upper) semicontinuous at $x_0 \in \mathbf{A}$. Show that $\sup\limits_{n\in\mathbb{N}} f_n$ (resp. $\inf\limits_{n\in\mathbb{N}} f_n$) is lower (resp. upper) semicontinuous at x_0.

1.4.18. Prove that a pointwise limit of an increasing (resp. decreasing) sequence of lower (resp. upper) semicontinuous functions is lower (resp. upper) semicontinuous.

1.4.19. For $f : \mathbf{A} \to \mathbb{R}$ and x a limit point of $\mathbf{A}$ define the *oscillation of f at x* by

$$o_f(x) = \lim_{\delta\to 0^+} \sup\{|f(z) - f(u)| : \ z, u \in \mathbf{A}, \ |z - x| < \delta, \ |u - x| < \delta\}$$

Show that $o_f(x) = f_1(x) - f_2(x)$, where

$$f_1(x) = \max\{f(x), \varlimsup_{z\to x} f(z)\} \quad \text{and} \quad f_2(x) = \min\{f(x), \varliminf_{z\to x} f(z)\}.$$

1.4.20. Let f_1, f_2, and o_f be as in the foregoing problem. Show that f_1 and o_f are upper semicontinuous, and f_2 is lower semicontinuous.

1.4.21. Prove that in order that $f : \mathbf{A} \to \mathbb{R}$ be lower (resp. upper) semicontinuous at $x_0 \in \mathbf{A}$, a necessary and sufficient condition is that for every $a < f(x_0)$ (resp. $a > f(x_0)$) there is $\delta > 0$ such that $f(x) > a$ (resp. $f(x) < a$) whenever $|x - x_0| < \delta$, $x \in \mathbf{A}$.

1.4.22. Prove that in order that $f : \mathbf{A} \to \mathbb{R}$ be lower (resp. upper) semicontinuous on $\mathbf{A}$, a necessary and sufficient condition is that for every $a \in \mathbb{R}$ the set $\{x \in \mathbf{A} : \ f(x) > a\}$ (resp. $\{x \in \mathbf{A} : \ f(x) < a\}$) be open in $\mathbf{A}$.

1.4.23. Prove that $f : \mathbb{R} \to \mathbb{R}$ is lower semicontinuous if and only if the set $\{(x, y) \in \mathbb{R}^2 : y \geq f(x)\}$ is closed in $\mathbb{R}^2$.

Formulate and prove an analogous necessary and sufficient condition for upper semicontinuity of f on $\mathbb{R}$.

1.4.24. Prove the following *theorem of Baire*. Every lower (resp. upper) semicontinuous $f : \mathbf{A} \to \mathbb{R}$ is the pointwise limit of an increasing (resp. decreasing) sequence of continuous functions on $\mathbf{A}$.

1.4.25. Prove that if $f : \mathbf{A} \to \mathbb{R}$ is upper semicontinuous, $g : \mathbf{A} \to \mathbb{R}$ is lower semicontinuous and $f(x) \le g(x)$ everywhere on $\mathbf{A}$, then there is a continuous function h on $\mathbf{A}$ such that

$$f(x) \le h(x) \le g(x), \quad x \in \mathbf{A}.$$

1.5. Uniform Continuity

Definition. A real function f defined on $\mathbf{A} \subset \mathbb{R}$ is said to be *uniformly continuous* on $\mathbf{A}$ if, given $\varepsilon > 0$, there exists $\delta > 0$ such that for all x and y in $\mathbf{A}$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$.

1.5.1. Verify whether the following functions are uniformly continuous on $(0, 1)$:

(a) $f(x) = e^x$, (b) $f(x) = \sin \dfrac{1}{x}$,

(c) $f(x) = x \sin \dfrac{1}{x}$, (d) $f(x) = e^{\frac{1}{x}}$,

(e) $f(x) = e^{-\frac{1}{x}}$, (f) $f(x) = e^x \cos \dfrac{1}{x}$,

(g) $f(x) = \ln x$, (h) $f(x) = \cos x \cdot \cos \dfrac{\pi}{x}$,

(i) $f(x) = \cot x$.

1.5.2. Which of the following functions are uniformly continuous on $[0, \infty)$?

(a) $f(x) = \sqrt{x}$, (b) $f(x) = x \sin x$,

(c) $f(x) = \sin^2 x$, (d) $f(x) = \sin(x^2)$,

(e) $f(x) = e^x$, (f) $f(x) = e^{\sin(x^2)}$,

(g) $f(x) = \sin(\sin x)$, (h) $f(x) = \sin(x \sin x)$,

(i) $f(x) = \sin \sqrt{x}$.

1.5.3. Show that if f is uniformly continuous on (a, b), $a, b \in \mathbb{R}$, then $\lim\limits_{x \to a^+} f(x)$ and $\lim\limits_{x \to b^-} f(x)$ exist as finite limits.

1.5.4. Suppose f and g are uniformly continuous on (a,b) ($[a,\infty)$). Does this imply the uniform continuity on (a,b) ($[a,\infty)$) of the functions

(a) $f+g$, (b) fg, (c) $x \mapsto f(x)\sin x$?

1.5.5.

(a) Show that if f is uniformly continuous on $(a,b]$ and on $[b,c)$, then it is also uniformly continuous on (a,c).

(b) Suppose $\mathbf{A}$ and $\mathbf{B}$ are closed sets in $\mathbb{R}$ and let $f : \mathbf{A} \cup \mathbf{B} \to \mathbb{R}$ be uniformly continuous on $\mathbf{A}$ and on $\mathbf{B}$. Must f be uniformly continuous on $\mathbf{A} \cup \mathbf{B}$?

1.5.6. Prove that any function continuous and periodic on $\mathbb{R}$ must be uniformly continuous on $\mathbb{R}$.

1.5.7.

(a) Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and such that $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to\infty} f(x)$ are finite, then f is uniformly continuous on $\mathbb{R}$.

(b) Show that if $f : [a,\infty) \to \mathbb{R}$ is continuous and $\lim\limits_{x\to\infty} f(x)$ is finite, then f is uniformly continuous on $[a,\infty)$.

1.5.8. Examine the uniform continuity of

(a) $f(x) = \arctan x$ on $(-\infty,\infty)$,

(b) $f(x) = x\sin\frac{1}{x}$ on $(0,\infty)$,

(c) $f(x) = e^{-\frac{1}{x}}$ on $(0,\infty)$.

1.5.9. Assume that f is uniformly continuous on $(0,\infty)$. Must the limits $\lim\limits_{x\to0^+} f(x)$ and $\lim\limits_{x\to\infty} f(x)$ exist?

1.5.10. Prove that any function which is bounded, monotonic and continuous on an interval $\mathbf{I} \subset \mathbb{R}$ is uniformly continuous on $\mathbf{I}$.

1.5.11. Assume f is uniformly continuous and unbounded on $[0,\infty)$. Is it true that either $\lim\limits_{x\to\infty} f(x) = +\infty$ or $\lim\limits_{x\to\infty} f(x) = -\infty$?

1.5.12. A function $f : [0,\infty) \to \mathbb{R}$ is uniformly continuous and for any $x \geq 0$ the sequence $\{f(x+n)\}$ converges to zero. Prove that $\lim\limits_{x\to\infty} f(x) = 0$.

1.5.13. Suppose that $f : [1,\infty) \to \mathbb{R}$ is uniformly continuous. Prove that there is a positive M such that $\frac{|f(x)|}{x} \leq M$ for $x \geq 1$.

1.5.14. Let $f : [0,\infty) \to \mathbb{R}$ be uniformly continuous. Prove that there is a positive M with the following property:

$$\sup_{u>0}\{|f(x+u) - f(u)|\} \leq M(x+1) \quad \text{for every} \quad x \geq 0.$$

1.5.15. Let $f : \mathbf{A} \to \mathbb{R}$, $\mathbf{A} \subset \mathbb{R}$, be uniformly continuous. Prove that if $\{x_n\}$ is any Cauchy sequence of elements in $\mathbf{A}$, then $\{f(x_n)\}$ is also a Cauchy sequence.

1.5.16. Suppose $\mathbf{A} \subset \mathbb{R}$ is bounded. Prove that if $f : \mathbf{A} \to \mathbb{R}$ transforms Cauchy sequences of elements of $\mathbf{A}$ into Cauchy sequences, then f is uniformly continuous on $\mathbf{A}$. Is the boundedness of $\mathbf{A}$ an essential assumption?

1.5.17. Prove that f is uniformly continuous on $\mathbf{A} \subset \mathbb{R}$ if and only if for any sequences $\{x_n\}$ and $\{y_n\}$ of elements of $\mathbf{A}$,

$$\lim_{n\to\infty}(x_n - y_n) = 0 \quad \text{implies} \quad \lim_{n\to\infty}(f(x_n) - f(y_n)) = 0.$$

1.5.18. Suppose that $f : (0,\infty) \to (0,\infty)$ is uniformly continuous. Does this imply that

$$\lim_{x\to\infty} \frac{f\left(x + \frac{1}{x}\right)}{f(x)} = 1?$$

1.5.19. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at zero and satisfies the following conditions

$$f(0) = 0 \quad \text{and} \quad f(x_1 + x_2) \leq f(x_1) + f(x_2) \quad \text{for any } x_1, x_2 \in \mathbb{R}.$$

Prove that f is uniformly continuous on $\mathbb{R}$.

1.5.20. For $f : \mathbf{A} \to \mathbb{R}$, $\mathbf{A} \subset \mathbb{R}$, we define

$$\omega_f(\delta) = \sup\{|f(x_1) - f(x_2)| : \quad x_1, x_2 \in \mathbf{A}, \ |x_1 - x_2| < \delta\}$$

and call ω_f the *modulus of continuity of f*. Show that f is uniformly continuous on $\mathbf{A}$ if and only if $\lim_{\delta \to 0^+} \omega_f(\delta) = 0$.

1.5.21. Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous. Prove that the following statements are equivalent.

(a) For any uniformly continuous function $g : \mathbb{R} \to \mathbb{R}$, $f \cdot g$ is uniformly continuous on $\mathbb{R}$.

(b) The function $x \mapsto |x|f(x)$ is uniformly continuous on $\mathbb{R}$.

1.5.22. Prove that the following condition is necessary and sufficient for f to be uniformly continuous on an interval $\mathbf{I}$. Given $\varepsilon > 0$, there is $N > 0$ such that for every $x_1, x_2 \in \mathbf{I}$, $x_1 \neq x_2$,

$$\left|\frac{f(x_1) - f(x_2)}{x_1 - x_2}\right| > N \quad \text{implies} \quad |f(x_1) - f(x_2)| < \varepsilon.$$

1.6. Functional Equations

1.6.1. Prove that the only functions continuous on $\mathbb{R}$ and satisfying the *Cauchy functional equation*

$$f(x + y) = f(x) + f(y)$$

are the linear functions of the form $f(x) = ax$.

1.6.2. Prove that if $f : \mathbb{R} \to \mathbb{R}$ satisfies the Cauchy functional equation

$$f(x + y) = f(x) + f(y)$$

and one of the conditions

(a) f is continuous at an $x_0 \in \mathbb{R}$,

(b) f is bounded above on some interval (a, b),

(c) f is monotonic on $\mathbb{R}$,

then $f(x) = ax$.

1.6.3. Determine all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(1) > 0$ and

$$f(x+y) = f(x)f(y).$$

1.6.4. Show that the only solutions of the functional equation

$$f(xy) = f(x) + f(y)$$

which are not identically zero and are continuous on $(0, \infty)$ are the logarithmic functions.

1.6.5. Show that the only solutions of the functional equation

$$f(xy) = f(x)f(y)$$

which are not identically zero and are continuous on $(0, \infty)$ are the power functions of the form $f(x) = x^a$.

1.6.6. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) - f(y)$ is rational for rational $x - y$.

1.6.7. For $|q| < 1$, find all functions $f : \mathbb{R} \to \mathbb{R}$ continuous at zero and satisfying the functional equation

$$f(x) + f(qx) = 0.$$

1.6.8. Find all functions $f : \mathbb{R} \to \mathbb{R}$ continuous at zero and satisfying the equation

$$f(x) + f\left(\frac{2}{3}x\right) = x.$$

1.6.9. Determine all solutions $f : \mathbb{R} \to \mathbb{R}$ of the functional equation

$$2f(2x) = f(x) + x$$

which are continuous at zero.

1.6.10. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the *Jensen equation*

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}.$$

1.6.11. Find all functions continuous on (a, b), $a, b \in \mathbb{R}$, satisfying the Jensen equation

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}.$$

1.6.12. Determine all solutions $f : \mathbb{R} \to \mathbb{R}$ of the functional equation

$$f(2x+1) = f(x)$$

which are continuous at -1.

1.6.13. For a real a, show that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous solution of the equation

$$f(x+y) = f(x) + f(y) + axy,$$

then $f(x) = \frac{a}{2}x^2 + bx$, where $b = f(1) - \frac{a}{2}$.

1.6.14. Determine all continuous at zero solutions of the functional equation

$$f(x) = f\left(\frac{x}{1-x}\right), \quad x \neq 1.$$

1.6.15. Let $f : [0,1] \to [0,1]$ be continuous, monotonically decreasing and such that $f(f(x)) = x$ for $x \in [0,1]$. Is $f(x) = 1 - x$ the only such function?

1.6.16. Suppose that f and g satisfy the equation

$$f(x+y) + f(x-y) = 2f(x)g(y), \quad x, y \in \mathbb{R}.$$

Show that if f is not identically zero and $|f(x)| \leq 1$ for $x \in \mathbb{R}$, then also $|g(x)| \leq 1$ for $x \in \mathbb{R}$.

1.6.17. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the functional equation

$$f(x+y) = f(x)e^y + f(y)e^x.$$

1.6.18. Determine all continuous at zero solutions $f : \mathbb{R} \to \mathbb{R}$ of

$$f(x+y) - f(x-y) = f(x)f(y).$$

1.6.19. Solve the functional equation

$$f(x) + f\left(\frac{x-1}{x}\right) = 1 + x \quad \text{for} \quad x \neq 0, 1.$$

1.6.20. A sequence $\{x_n\}$ *converges in the Cesàro sense* if

$$C\text{-}\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$

exists and is finite. Find all functions which are *Cesàro continuous,* that is,

$$f(C\text{-}\lim_{n\to\infty} x_n) = C\text{-}\lim_{n\to\infty} f(x_n)$$

for every Cesàro convergent sequence $\{x_n\}$.

1.6.21. Let $f : [0,1] \to [0,1]$ be an injection such that $f(2x - f(x)) = x$ for $x \in [0,1]$. Prove that $f(x) = x$, $x \in [0,1]$.

1.6.22. For m different from zero, prove that if a continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfies the equation

$$f\left(2x - \frac{f(x)}{m}\right) = mx,$$

then $f(x) = m(x - c)$.

1.6.23. Show that the only solutions of the functional equation

$$f(x + y) + f(y - x) = 2f(x)f(y)$$

continuous on $\mathbb{R}$ and not identically zero are $f(x) = \cos(ax)$ and $f(x) = \cosh(ax)$ with a real.

1.6.24. Determine all continuous on $(-1, 1)$ solutions of

$$f\left(\frac{x+y}{1+xy}\right) = f(x) + f(y).$$

1.6.25. Find all polynomials P such that

$$P(2x - x^2) = (P(x))^2.$$

1.6.26. Let $m, n \geq 2$ be integers. Find all functions $f : [0, \infty) \to \mathbb{R}$ continuous at at least one point in $[0, \infty)$ and such that

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^m\right) = \frac{1}{n}\sum_{i=1}^{n}(f(x_i))^m \quad \text{for} \quad x_i \geq 0,\ i = 1, 2, \ldots, n.$$

1.6.27. Find all not identically zero functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equations

$$f(xy) = f(x)f(y) \quad \text{and} \quad f(x+z) = f(x) + f(z)$$

with some $z \neq 0$.

1.6.28. Find all functions $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ such that

$$f(x) = -f\left(\frac{1}{x}\right), \quad x \neq 0.$$

1.6.29. Find all solutions $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ of the functional equation

$$f(x) + f(x^2) = f\left(\frac{1}{x}\right) + f\left(\frac{1}{x^2}\right), \quad x \neq 0.$$

1.6.30. Prove that the functions $f, g, \phi : \mathbb{R} \to \mathbb{R}$ satisfy the equation

$$\frac{f(x) - g(y)}{x - y} = \phi\left(\frac{x+y}{2}\right), \quad y \neq x,$$

if and only if there exist a, b and c such that

$$f(x) = g(x) = ax^2 + bx + c, \quad \phi(x) = 2ax + b.$$

1.6.31. Prove that there is a function $f : \mathbb{R} \to \mathbb{Q}$ satisfying the following three conditions:

(a) $f(x+y) = f(x) + f(y)$ for $x, y \in \mathbb{R}$,

(b) $f(x) = x$ for $x \in \mathbb{Q}$,

(c) f is not continuous on $\mathbb{R}$.

1.7. Continuous Functions in Metric Spaces

In this section $\mathbf{X}$ and $\mathbf{Y}$ will stand for metric spaces $(\mathbf{X}, d_1)$ and $(\mathbf{Y}, d_2)$, respectively. To shorten notation we say that $\mathbf{X}$ is a metric space instead of saying that $(\mathbf{X}, d_1)$ is a metric space. If not stated otherwise, $\mathbb{R}$ and $\mathbb{R}^n$ are always assumed to be equipped with the Euclidean metric.

1.7.1. Let $(\mathbf{X}, d_1)$ and $(\mathbf{Y}, d_2)$ be metric spaces and let $f : \mathbf{X} \to \mathbf{Y}$. Prove that the following conditions are equivalent.

(a) The function f is continuous.

(b) For each closed set $\mathbf{F} \subset \mathbf{Y}$ the set $f^{-1}(\mathbf{F})$ is closed in $\mathbf{X}$.

(c) For each open set $\mathbf{G} \subset \mathbf{Y}$ the set $f^{-1}(\mathbf{G})$ is open in $\mathbf{X}$.

(d) For each subset $\mathbf{A}$ of $\mathbf{X}$, $f(\overline{\mathbf{A}}) \subset \overline{f(\mathbf{A})}$.

(e) For each subset $\mathbf{B}$ of $\mathbf{Y}$, $\overline{f^{-1}(\mathbf{B})} \subset f^{-1}(\overline{\mathbf{B}})$.

1.7.2. Let $(\mathbf{X}, d_1)$ and $(\mathbf{Y}, d_2)$ be metric spaces and let $f : \mathbf{X} \to \mathbf{Y}$ be continuous. Prove that the inverse image $f^{-1}(\mathbf{B})$ of a Borel set $\mathbf{B}$ in $(\mathbf{Y}, d_2)$ is a Borel set in $(\mathbf{X}, d_1)$.

1.7.3. Give an example of a continuous function $f : \mathbf{X} \to \mathbf{Y}$ such that the image $f(\mathbf{F})$ (resp. $f(\mathbf{G})$) is not closed (resp. open) in $\mathbf{Y}$ for a closed $\mathbf{F}$ (resp. open $\mathbf{G}$) in $\mathbf{X}$.

1.7.4. Let $(\mathbf{X}, d_1)$ and $(\mathbf{Y}, d_2)$ be metric spaces and let $f : \mathbf{X} \to \mathbf{Y}$ be continuous. Prove that the image of each compact set $\mathbf{F}$ in $\mathbf{X}$ is compact in $\mathbf{Y}$.

1.7.5. Let f be defined on the union of closed sets $\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_m$. Prove that if the restriction of f to each $\mathbf{F}_i$, $i = 1, 2, \dots, m$, is continuous, then f is continuous on $\mathbf{F}_1 \cup \mathbf{F}_2 \cup \dots \cup \mathbf{F}_m$.

Show by example that the statement does not hold in the case of infinitely many sets $\mathbf{F}_i$.

1.7.6. Let f be defined on the union of open sets $\mathbf{G}_t$, $t \in \mathbf{T}$. Prove that if for each $t \in \mathbf{T}$ the restriction $f_{|\mathbf{G}_t}$ is continuous, then f is continuous on $\bigcup\limits_{t \in \mathbf{T}} \mathbf{G}_t$.

1.7.7. Let $(\mathbf{X}, d_1)$ and $(\mathbf{Y}, d_2)$ be metric spaces. Prove that $f : \mathbf{X} \to \mathbf{Y}$ is continuous if and only if for each compact $\mathbf{A}$ in $\mathbf{X}$ the function $f_{|\mathbf{A}}$ is continuous.

1.7.8. Assume that f is a continuous bijection of a compact metric space $\mathbf{X}$ onto a metric space $\mathbf{Y}$. Prove that the inverse function f^{-1} is continuous on $\mathbf{Y}$. Prove also that compactness cannot be omitted from the hypotheses.

1.7.9. Let f be a continuous mapping of a compact metric space $\mathbf{X}$ into a metric space $\mathbf{Y}$. Show that f is uniformly continuous on $\mathbf{X}$.

1.7.10. Let $(\mathbf{X}, d)$ be a metric space and let $\mathbf{A}$ be a nonempty subset of $\mathbf{X}$. Prove that the function $f : \mathbf{X} \to [0, \infty)$ defined by

$$f(x) = \operatorname{dist}(x, \mathbf{A}) = \inf\{d(x, y) : \ y \in \mathbf{A}\}$$

is uniformly continuous on $\mathbf{X}$.

1.7.11. Assume that f is a continuous mapping of a connected metric space $\mathbf{X}$ into a metric space $\mathbf{Y}$. Show that $f(\mathbf{X})$ is connected in $\mathbf{Y}$.

1.7.12. Let $f : \mathbf{A} \to \mathbf{Y}$, $\emptyset \neq \mathbf{A} \subset \mathbf{X}$. For $x \in \overline{\mathbf{A}}$ define

$$o_f(x, \delta) = \operatorname{diam}(f(\mathbf{A} \cap \mathbf{B}(x, \delta))).$$

The *oscillation of f at x* is defined as

$$o_f(x) = \lim_{\delta \to 0^+} o_f(x, \delta).$$

Prove that f is continuous at $x_0 \in \mathbf{A}$ if and only if $o_f(x_0) = 0$ (compare with 1.4.19 and 1.4.20).

1.7.13. Let $f : \mathbf{A} \to \mathbf{Y}$, $\emptyset \neq \mathbf{A} \subset \mathbf{X}$ and for $x \in \overline{\mathbf{A}}$ let $o_f(x)$ be the oscillation of f at x defined in the foregoing problem. Prove that for each $\varepsilon > 0$ the set $\{x \in \overline{\mathbf{A}} : \ o_f(x) \geq \varepsilon\}$ is closed in $\mathbf{X}$.

1.7.14. Show that the set of points of continuity of $f : \mathbf{X} \to \mathbf{Y}$ is a countable intersection of open sets, that is, a $\mathcal{G}_\delta$ in $(\mathbf{X}, d_1)$. Show also that the set of points of discontinuity of f is a countable union of closed sets, that is, an $\mathcal{F}_\sigma$ in $(\mathbf{X}, d_1)$.

1.7.15. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ whose set of points of discontinuity is $\mathbb{Q}$.

1.7.16. Prove that every $\mathcal{F}_\sigma$ subset of $\mathbb{R}$ is the set of points of discontinuity for some $f : \mathbb{R} \to \mathbb{R}$.

1.7.17. Let $\mathbf{A}$ be an $\mathcal{F}_\sigma$ subset of a metric space $\mathbf{X}$. Must there exist a function $f : \mathbf{X} \to \mathbb{R}$ whose set of points of discontinuity is $\mathbf{A}$?

1.7.18. Let $\chi_\mathbf{A}$ be the characteristic function of $\mathbf{A} \subset \mathbf{X}$. Show that $\{x \in \mathbf{X} : o_{\chi_\mathbf{A}}(x) > 0\} = \partial\mathbf{A}$, where $o_f(x)$ is the oscillation of f at x defined in 1.7.12. Conclude that $\chi_\mathbf{A}$ is continuous on $\mathbf{X}$ if and only if $\mathbf{A}$ is both open and closed in $\mathbf{X}$.

1.7.19. Assume that g_1 and g_2 are continuous mappings of a metric space $(\mathbf{X}, d_1)$ into a metric space $(\mathbf{Y}, d_2)$, and that a set $\mathbf{A}$ with a void interior is dense in $\mathbf{X}$. Prove that if

$$f(x) = \begin{cases} g_1(x) & \text{for} \quad x \in \mathbf{A}, \\ g_2(x) & \text{for} \quad x \in \mathbf{X} \setminus \mathbf{A}, \end{cases}$$

then

$$o_f(x) = d_2(g_1(x), g_2(x)), \quad x \in \mathbf{X},$$

where $o_f(x)$ is the oscillation of f at x defined in 1.7.12.

1.7.20. We say that a real function f defined on a metric space $\mathbf{X}$ is in the *first Baire class* if f is a pointwise limit of a sequence of continuous functions on $\mathbf{X}$. Prove that if f is in the first Baire class, then the set of points of discontinuity of f is a set of the first category; that is, it is the union of countably many nowhere dense sets.

1.7.21. Prove that if $\mathbf{X}$ is a complete metric space and f is in the first Baire class on $\mathbf{X}$, then the set of points of continuity of f is dense in $\mathbf{X}$.

1.7.22. Let $f : (0, \infty) \to \mathbb{R}$ be continuous and such that, for each positive x, the sequence $\left\{f\left(\frac{x}{n}\right)\right\}$ converges to zero. Does this imply that $\lim\limits_{x \to 0^+} f(x) = 0$? (Compare with 1.1.33.)

1.7.23. Let $\mathcal{F}$ denote a family of real functions continuous on a complete metric space $\mathbf{X}$ such that for every $x \in \mathbf{X}$ there is M_x such that

$$|f(x)| \le M_x \quad \text{for all } f \in \mathcal{F}.$$

Prove that there exist a positive constant M and a nonempty open set $\mathbf{G} \subset \mathbf{X}$ such that

$$|f(x)| \le M \quad \text{for every } f \in \mathcal{F} \quad \text{and every } x \in \mathbf{G}.$$

1.7.24. Let $\mathbf{F}_1 \supset \mathbf{F}_2 \supset \mathbf{F}_3 \supset \ldots$ be a nested collection of nonempty closed subsets of a complete metric space $\mathbf{X}$ such that $\lim\limits_{n\to\infty} \operatorname{diam} \mathbf{F}_n = 0$. Prove that if f is continuous on $\mathbf{X}$, then

$$f\left(\bigcap_{n=1}^{\infty} \mathbf{F}_n\right) = \bigcap_{n=1}^{\infty} f(\mathbf{F}_n).$$

1.7.25. Let $(\mathbf{X}, d_1)$ be a metric space and p a fixed point in $\mathbf{X}$. For $u \in \mathbf{X}$ define the function f_u by $f_u(x) = d_1(u,x) - d_1(p,x)$, $x \in \mathbf{X}$. Prove that $u \mapsto f_u$ is a distance preserving mapping, that is, an isometry of $(\mathbf{X}, d_1)$ into the space $C(\mathbf{X}, \mathbb{R})$ of real functions continuous on $\mathbf{X}$ endowed with the metric $d(f,g) = \sup\{|f(x) - g(x)| : x \in \mathbf{X}\}$.

1.7.26. Prove that a metric space $\mathbf{X}$ is compact if and only if every continuous function $f : \mathbf{X} \to \mathbb{R}$ is bounded.

1.7.27. Let $(\mathbf{X}, d_1)$ be a metric space and for $x \in \mathbf{X}$ define $\rho(x) = \operatorname{dist}(x, \mathbf{X} \setminus \{x\})$. Prove that the following two conditions are equivalent.

(a) Each continuous function $f : \mathbf{X} \to \mathbb{R}$ is uniformly continuous.

(b) Every sequence $\{x_n\}$ of elements in $\mathbf{X}$ such that

$$\lim_{n\to\infty} \rho(x_n) = 0$$

contains a convergent subsequence.

1.7.28. Show that a metric space $\mathbf{X}$ is compact if and only if every real function continuous on $\mathbf{X}$ is uniformly continuous and for every $\varepsilon > 0$ the set $\{x \in \mathbf{X} : \rho(x) > \varepsilon\}$, where ρ is defined in 1.7.27, is finite.

1.7.29. Give an example of a noncompact metric space $\mathbf{X}$ such that every continuous $f : \mathbf{X} \to \mathbb{R}$ is uniformly continuous on $\mathbf{X}$.