We leave it to the reader to show that if $p$ is any polynomial, then

$$\lim_{x\to 0} p(1/x)e^{-1/x^2} = 0.$$

(Hint: $e^x > x^n/n!$ for every $x > 0$.)

From this fact, it follows easily that $f$ is infinitely differentiable and that $f^{(n)}(0) = 0$, $n = 1, 2, \cdots$. The Taylor series of $f$, with $a = 0$, thus converges to the zero function, not to $f$.

Chapter 9 will be devoted entirely to a study of series of the form $\sum_{n=0}^{\infty} a_n x^n$. Such series are called power series.

## EXERCISES

**1.1** Give an example of a continuous function which is not differentiable on a dense set.

**2.1** If $f$ and $g$ are $n$ times differentiable, obtain a formula for the $n$th derivative of $fg$.

**2.2** If $f$ is differentiable and not 0, show that

$$\left(\frac{1}{f}\right)' = -\frac{f'}{f^2}.$$

**2.3** If $f$ and $g$ are infinitely differentiable, show that $f \circ g$ is infinitely differentiable.

**2.4** If $x = \cos^3 t$ and $y = t \sin^3 t$, find

$$\frac{d^2y}{dx^2}.$$

**3.1** Verify that the derivative of

$$f(x) = \begin{cases} x^2 \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is

$$f'(x) = \begin{cases} 2x \sin 1/x - \cos 1/x, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

**3.2** The right derivative of $f$ exists at $x_0$ and is equal to $k$ if, for every $\epsilon > 0$, there is a $\delta > 0$ such that $0 < x - x_0 < \delta$ implies that $|[f(x) - f(x_0)]/(x - x_0) - k| < \epsilon$. The left derivative is defined similarly. Show that the set of points where the left and right derivatives both exist, but are not equal, is countable.

**3.3** If $\lim_{h\to 0} [f(x + h) + f(x - h) - 2f(x)]/h^2$ exists, then $f$ is said to have a generalized second derivative at $x$ which is given by the above limit. Show that if $f$ has a generalized second derivative at $x$, then it has a second derivative at $x$, and the two are equal.

**3.4** Give an example of a function which does not have a second derivative at a point but does have a generalized second derivative there.

**3.5** If $f$ is twice differentiable, and the second derivative is never negative, show that $f$ is convex.

**3.6** Give an example of a convex function whose derivative does not exist on a dense set.

**3.7** If $f$ is convex, show that its right and left derivatives exist everywhere and are nondecreasing.

**3.8** Given an arbitrary countable set $S$, give an example of a convex function whose derivative does not exist at any point in $S$ but does exist at every point not in $S$.

**3.9** If a function is continuous on $[a, b]$, and its generalized second derivative is zero everywhere on $[a, b]$, show that the function is linear on $[a, b]$.

**3.10** Prove the generalized mean value theorem which says that if $f$ and $g$ are continuous on $[a, b]$, and differentiable on $(a, b)$, there is a $\xi \in (a, b)$ such that

$$f'(\xi)[g(b) - g(a)] = g'(\xi)[f(b) - f(a)].$$

**4.1** Give an example of a series of continuously differentiable functions which converges uniformly but whose derivatives do not converge.

**4.2** Find log 1.3 to 4 decimals, and prove your answer correct.

**4.3** Find $\sqrt[3]{7}$ to 6 decimals, and prove your answer correct.

**5.1** Show that if $p$ is any polynomial, then

$$\lim_{x\to 0} p(1/x)e^{-1/x^2} = 0.$$

**5.2** Show that the function

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is infinitely differentiable.

**5.3** If $(a, b)$ is an open interval, give an example of an infinitely differentiable function which is positive on $(a, b)$ and zero everywhere else.

**5.4** For every continuous function $f$ on the reals, show that there is a sequence of infinitely differentiable functions which converges uniformly to $f$.

**5.5** Give an example of a function which has the Darboux property, but has at most one point of continuity.

**5.6** If $f$: $[a, b] \to [c, d]$ is bijective, differentiable, with continuous derivative which is always positive, then the inverse function has the same property.