EXERCISES

2.1 If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series of positive numbers, show that the series $\sum_{n=1}^{\infty} a_n^{1/2} b_n^{1/2}$ converges.

2.2 If $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive numbers, and $p > 1$, show that $\sum_{n=1}^{\infty} a_n^p$ converges.

2.3 Give an example of a convergent series $\sum_{n=1}^{\infty} a_n$ of positive terms, such that the series $\sum_{n=1}^{\infty} a_n^p$ diverges for every p, with $0 < p < 1$.

2.4 Given a sequence of series of positive terms, each of which converges, show that there is a series which converges more slowly than all of them.

2.5 Given an example of a series of positive terms which converges more slowly than all the series

$$\frac{1}{n \log n \log \log n \cdots \underbrace{(\log \cdots \log n)}_{m}^{2}},$$

$m = 1, 2, \cdots$.

2.6 Show that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive terms which are decreasing, then $\lim_n n a_n = 0$.

2.7 Show that the converse of Exercise 2.6 does not hold.

2.8 If $\sum_{n=1}^{\infty} a_n$ is a series of decreasing positive terms, and $\sum_{n=1}^{\infty} (a_n a_{n+1})^{1/2}$ converges, show that $\sum_{n=1}^{\infty} a_n$ converges.

2.9 An infinite product $\prod_{n=1}^{\infty} (1 + a_n)$, $a_n > 0$, $n = 1, 2, \cdots$, is said to be convergent if the sequence $s_n = \prod_{k=1}^{n} (1 + a_k)$ converges. Show that the infinite product converges if and only if the infinite series $\sum_{n=1}^{\infty} a_n$ converges.

2.10 For what values of p does the series

$$1 + \left(\frac{1}{2}\right)^p + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^p + \cdots$$

converge?

2.11 Generalize the result of Exercise 2.10.

2.12 If $\sum_{n=1}^{\infty} a_n$ is an infinite series of positive terms, show that it converges if

$$\liminf_n \left[n\left(\frac{a_n}{a_{n+1}} - 1\right)\right] > 1.$$

2.13 Show that it diverges if

$$\limsup_n \left[n\left(\frac{a_n}{a_{n+1}} - 1\right)\right] < 1.$$

2.14 Give an example which cannot be treated by the results of Exercises 2.12 and 2.13.

2.15 Does the series

$$\frac{1}{\log 2} + \frac{1}{\log 3} + \cdots + \frac{1}{\log n} + \cdots$$

converge or diverge?

2.16 Does the series $\sum_{n=1}^{\infty} \dfrac{e^n n!}{n^n}$ converge or diverge?

2.17 If $\sum_{n=1}^{\infty} a_n^2 < \infty$ and $\sum_{n=1}^{\infty} b_n^2 < \infty$, show that

$$\left(\sum_{n=1}^{\infty} a_n b_n\right)^2 \leq \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2.$$

2.18 If $\sum_{n=1}^{\infty} a_n^2 < \infty$ and $\sum_{n=1}^{\infty} b_n^2 < \infty$ show that

$$\left[\sum_{n=1}^{\infty} (a_n + b_n)^2\right]^{1/2} \leq \left[\sum_{n=1}^{\infty} a_n^2\right]^{1/2} + \left[\sum_{=1n}^{\infty} b_n^2\right]^{1/2}.$$

3.1 If $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ are given, with $s_k = a_1 + \cdots + a_k$, $k = 1, \cdots, n$, show that

$$\sum_{k=1}^{\infty} a_k b_k = \sum_{n=1}^{\infty} s_{k-1}(b_{k-1} - b_k) + s_n b_n.$$

3.2 If the sequence $\{s_n\}$ of partial sums of $\sum_{n=1}^{\infty} a_n$ is bounded and the sequence $\{b_n\}$ is nonincreasing and converges to zero, then the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

3.3 If $\sum_{n=1}^{\infty} a_n$ converges and $\{b_n\}$ is nonincreasing, show that $\sum_{n=1}^{\infty} a_n b_n$ converges.

3.4 If the terms of a series alternate in sign, decrease in absolute value, and converge to zero, show that the series converges.

3.5 Given any closed interval $[a, b]$, show that every conditionally convergent series has a rearrangement which has $[a, b]$ as the set of limit points of its sequence of partial sums.

3.6 Give an example of a series for which $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} a_n^2$ diverges.

3.7 If every subseries of a series converges, show that the series converges absolutely.

4.1 If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are series, its Cauchy product is defined as the series $\sum_{n=0}^{\infty} c_n$, where

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0.$$

Give an example of two divergent series whose Cauchy product converges.

4.2 If $\sum_{n=1}^{\infty} a_n$ converges to A, $\sum_{n=0}^{\infty} b_n$ converges to B, and $\sum_{n=0}^{\infty} a_n$ converges absolutely, show that their Cauchy product converges to AB.

4.3 If $\sum_{n=1}^{\infty} a_n = s$, $\sum_{n=0}^{\infty} b_n = t$, and their Cauchy product $\sum_{n=1}^{\infty} c_n$ converges, show that $\sum_{n=0}^{\infty} c_n = st$.